# TABLE OF CONTENTS

1 Abstract
2 Introduction
3 Choices
4 Ratios
5 Consonance and Dissonance
6 Irrational Ratios
8 Ratio Sequences
9 Two Tuning Methods
10 Asymmetry
12 Prime Limits
14 The Matrix and other Maps
15 Numerosity
17 The Musical Means
19 The Progression of Means
22 Another vision of the Genera
23 The Geometric Mean
25 Symmetry
27 The Classification of Harmony
30 Circles of Fifths
32 The Schisma
33 The City Wall
36 Aristoxenus and counter-arguments
40 Tetrachords
43 Planets
46 Octave Equivalence
47 Tolerance
51 Philosophical Foundations
54 The Three Ways and Universal Mixture
56 Inter-dimensionality
60 Inter-complimentarity
63 The Great Monochord Ritual
68 An Uncertain History
74 Compromise and Judgement
76 Summary
78 Afterwords

Charts and Diagrams (76 pages)
THE LAWS OF HARMONICS

Siemen Terpstra

ABSTRACT
The author explains the concepts, issues and underlying principles surrounding musical tuning.

INTRODUCTION

Starting in the middle 1980's, around the time that I wrote up *A Short List of Musical-Cosmological Monochords*, I began a patient compilation of diagrams. Most of them consisted of elaborate tables showing numerous just ratios calculated in cents, as well as some speculative monochords and representations of the Harmonic Series. I meant this work to accompany a text that sums up my investigations into Harmonics. Inevitably, it reflected my interests during the seventies and early eighties—just intonation, drone-based music, and monochords. I never got around to writing the text, though I left a couple of pages of notes. After a few years I scrapped the whole project and archived it under the rather imposing heading: *The Laws of Harmonics*. Recently I dragged it out of its storage to reconsider the project.

What should I do with this material? Several options came to mind. I could shelve it permanently, but then I had put so much effort into the calculations. Or I could write a text reflecting my positions in the mid-eighties, but that seems a rather peculiar 'historical' approach. Or I could write a text that proves true to the title. This path I have chosen. In this context the charts serve mainly as a mine or source of ratios to illustrate various principles. It works out quite well, even though in a few spots I must explain my old position and how it has mutated since then. I have found only one blatant change of terminology. The ratio tables live again!

I had shelved the project because of competing interests—a study of Aristoxenus, the features of multiple divisions, meantone tunings, irregular temperaments, ongoing keyboard and fretboard projects, and long investigations into early Greek philosophy. The last project in particular drove me deeper into the subtle aspects of Harmonics, forcing a change of perspective. Ancient philosophy is not the topic of this essay, and it gets only side-glances and short references. Yet it proves impossible to eliminate philosophy altogether from the subject of Harmonics, since they have so many overlapping concerns.

I have decided to write an essay of interest to the layman, rather than something restricted to tuning experts. Consequently, I try to use plain language, clear explanations, and a minimum of technical terms. Of course, this aim is somewhat of a challenge, since Harmonics is above all a 'numbers game.' Indeed, this essay is full of numbers, ratios, and monochord sequences. But I don’t expect any expertise in these matters. The numbers just form illustrations of some principle or issue. It's the principle that counts. Some terminology is unavoidable and some of it I had to coin myself. Needless to say, terminology must be clearly explained. I have also avoided footnotes and scholarly references, in order to appeal to non-specialists.
CHOICES

People like myself, who play stringed instruments, have often had this experience: We come together with other string players to make music, but first we must make various adjustments in order to get into tune with one another. Nowadays everyone uses various electronic devices as a help mate, but these gadgets are only a recent phenomenon. In the old days it took a subtle technique consisting of intuition, habit, careful listening and subliminal clues. Naturally we had good days and bad days, but eventually we would reach a point that was "close enough" to the ideal. The ideal is something we strive for, but the reality is always a little messier, a little deviated. Yet it works because we all share the same ideal as a reference point.

Many musicians don't realize that our modern European tuning ideal, twelve-equal-temperament (or 12-et) has only been dominant since the 19th century. Before that time a hodge-podge of different ideals held sway—various forms of well temperament and meantone temperament. Different corners of Europe had different tuning traditions, and certain instruments even had their own associated tunings. Makers of wind instruments had their own notions concerning scales and standards of pitch. Indeed, there simply was no universal standard pitch before the latter 19th century. Various theorists proposed tunings that may or may not even have been used.

Many modern music schools teach that the historical tunings are primitive, or defective, or merely steps in an evolution toward the 'correct' modern ideal. But these teachings are simply untrue. The historical tunings rather represent different aesthetic choices. Each choice has its consequences, its own particular flavor, its own pluses and minuses. Each choice can serve as an alternative ideal to strive for. Tuning norms are to some extent dependent on cultural biases.

Historical Europe, at least from the renaissance until recent times, can be described as pluralistic. No one tuning norm dominates the scene. Instead, many alternatives are entertained and exploited for their musical effects. Various controversies play out regarding preferences. Moreover, different choices had a real impact on musical styles and syntax.

Nor is Europe unique in its pluralism. It forms the norm in the exceedingly old classical tuning cultures of India, Iran, Turkey and the Middle-east. For example, in the Indian raga system various alternatives can be defined, all different from each other, though they prove to be related to each other. Each tuning has its own character or ethos, its own associated stock of melodic formulae, often even its own associated rhythmic patterns and performance contexts. Tuning is only one aspect of a complex set of parameters. This complexity is the signature of a sophisticated music culture.

We have ancient texts from India and China that give us important clues about their tuning traditions. Alas, no such texts have survived from ancient Babylonia or Egypt. Perhaps they never wrote anything, although I doubt it, since their music cultures
were also old and complex. They must have pondered musical tuning. At any rate ancient Greece holds great fascination because a whole group of creative writers discussed the issues over a period of almost one thousand years. Only a small fraction of these writings have survived, but it is enough to give us much of our terminology, our concepts, and even the controversies that resurface from time to time. And yes, classical Greece was also pluralistic.

Ancient Greece already had a complex music culture in the seventh century BC. Late in the sixth century Lassus of Hermione wrote about musical tuning, but his book is entirely lost. The oldest surviving fragments on the subject came from Philolaus around the late fifth century (the time of Socrates). However, Philolaus attributed his approach to Pythagoras, a contemporary of Lassus in the later sixth century. Early in the fourth century we witness a major figure, Archytas. The writing reached a climax in the late fourth century with Aristoxenus, arguably the greatest (and most controversial) of all the ancient tuning theorists. After him came a group of writers of secondary stature. In the third century Euclid defended Philolaus and attacked Aristoxenus. The Hellenistic writers Eratosthenes and Didymus showed progressive tendencies. Many more names can be cited, but little is actually new. At last, in the middle of the second century AD we find a giant figure to rival Aristoxenus. Ptolemy lived in Alexandria (Egypt) and made a synthesis of Greek and Middle-eastern science. After Ptolemy we find only lesser names, like Boethius in the early sixth century. Thus the two greatest names among the ancient theorists are Aristoxenus and Ptolemy, followed by Archytas and Philolaus.

All the surviving writings prove to be intensely polemical. Ptolemy, for example, defends his approach against that of Aristoxenus and the conservatives. Thus we can recognize several different schools or philosophies of musical tuning culture. Moreover, these schools persisted for hundreds of years in competition with one another. The issues and controversies raised have echoed throughout the later history. The next truly great tuning theorist was al-Farabi in tenth century Baghdad. He extended the approach of Ptolemy but managed to let it be influenced by aspects of Aristoxenus. In spite of al-Farabi’s brilliant efforts at reconciliation, Ptolemy and Aristoxenus have continued to be rivals who still have their apologists even today.

In order to understand why we have a plurality of choices and why this situation leads to controversy, we must delve into musical ratios and their peculiar properties.

RATIOS

What is a harmony in its simplest manifestation? We call it a musical interval. The closest such harmony is the ratio 1/1, the musical unison. It maximizes the power of resonance, a property of vibrating systems based on the reinforcement or prolongation of sound. It is caused by standing waves built in a tuned cavity or other vibrating system. Two guitar strings tuned to a unison form a highly sympathetic vibration, an expression of interconnected unity. One step lower in status sits the ratio 2/1, the octave. The power of resonance continues to drain away as ratio numbers grow, until complexity neutralizes it.
Seeking to define harmony, it's hard to improve on the definition given in early Greek philosophy (Philolaus). A *harmonikos* is a joining or relationship between unlike entities or elements, i.e. numbers. As such the first 'official' harmony is the octave ratio $2/1$. Indeed, one of the Greek names for the octave is simply *harmonikos*. The philosophers conceptualized the $1/1$ as a state of identity or unity (the One). The Greek term for ratio, *logos*, has always been closely associated with *harmonikos*. Their meanings overlap, with *logos* putting the emphasis on 'relation' and *harmonikos* on the 'unlike elements.' A harmony is thus a state of order, agreement, or aesthetic relationship among the elements of a whole. This relation can be consonant or dissonant, involving simple numbers or higher numbers, but as long as some ratio is present we can speak of a harmony.

We are confronted with a vibratory event that has psychoacoustical properties. It is at once the sound of an octave and the ratio $2/1$. The ultimate referent for the ratio is apparently the Harmonic Series (or so-called Overtone Series). Number relations specify vibratory events that are time dependent and that we hear as musical intervals. The language of numbers and the language of tuning theory collide.

A musical interval is a periodic or cyclical vibration that possesses a Period—its time duration in seconds, and a Frequency—the number of vibrations per second. Period and Frequency stand in inverse relation to each other. From this fundamental principle of physics comes the reciprocity of frequency and wavelength. As we will see, reciprocity is built into the fabric of harmony in various contexts. The relation between the numbers of a ratio is divisive/multiplicative. One can always represent a ratio in two ways—oriented up in pitch ($2/1$) or down in pitch ($1/2$). The principle of reciprocity resurfaces throughout my essay.

One can accurately define the study of Harmonics as the branch of arithmetic exploring how numbers relate to each other by multiplication and division. It comes out of the time dependency of vibratory events. Additive relations between numbers are irrelevant. Since late antiquity various writers have misappropriated the language of 'Harmonics' and 'vibrations,' referring it to numerological operations. For example, they commonly contend that the 'essence' of the number $17$ consists of a $1+7=8$ 'vibration,' generally a number between one and ten. But this is a fraud, either meant to deceive or to display ignorance. The true harmonic significance of the number $17$ comes from its status as a prime number, and from how it behaves in relation to other numbers in the context of ratios. *Ratio rules.*

Ratios are individuals, each with its own morphological characteristics and psychoacoustical features. Moreover, they place themselves on a bipolar Spectrum of *Consomance*. At one end sits utter simplicity and maximum consonance—the ratio $1/1$. Now the numbers grow and move into complexity and dissonance as we progress toward the other end of the Spectrum. But where is that end? Numbers proceed toward potential infinity, called by the ancients the *Unlimited*. Musical tuning imposes some pattern of *Limit*, some specific exemplar that 'kills' the potential infinite, yet also represents it in
actual time. There's always another ratio, but one quickly realizes that only a relatively small group of ratios (those with small numbers) embody consonance. The vast majority proves to be dissonant. The problem of the music tuner has always been the making of a scale that maximizes or at least balances out the inherent consonant elements and keeps the dissonances at bay. Different strategies have evolved for dealing with this problem.

CONSONANCE AND DISSONANCE

The complexity end of the Spectrum has always been difficult to deal with. It was a major preoccupation of early Greek philosophy. How should we model infinite divisibility? However, the other end of the Spectrum is clear enough. It begins in strong consonance and then eventually it fades away into dissonance. At the top of the heap sits 1/1 (unison). Then comes 2/1 (octave), then 3/1 (twelfth), 3/2 (fifth), 4/1 (double octave) and 4/3 (fourth). These ratios are often called the 'perfect consonances.' Then comes the group 5/1 (double octave plus major third), 5/2 (octave plus major third), 5/3 (major sixth), 5/4 (major third), 6/1 (octave plus twelfth), and 6/5 (minor third). These consonances are variously called medial, or tertiary. They are followed by a group whose consonance is decidedly more controversial. We are surely approaching the boundary with dissonance. Here one finds 7/1 (double octave plus augmented sixth), 7/2 (octave plus augmented sixth), 7/3 (octave plus augmented second), 7/4 (augmented sixth), 7/5 (augmented fourth), 7/6 (augmented second), 8/1 (triple octave), 8/3 (octave plus fourth), 8/5 (minor sixth), and 8/7 (stretched wholetone). These intervals are often called 'imperfect consonances' or near dissonances. Perhaps they are better called assonances—something between consonance and dissonance.

The criterion for consonance depends upon the clouding of Difference Tones in the timbre and other technical matters that need not concern us here. What is important is the observation that 8/7 sits 'on the fence' as it were. Further ratios, such as 9/8 and 10/9 (alternative wholetones) are judged to be soft dissonances or 'almost consonances,' again a sort of assonance. Moving further to 12/11 (three-quarter tone) one already encounters alien territory for European tuners, but this ratio is highly popular in Middle-eastern cultures. A ratio such as 16/15 (large semitone) is definitely heard as a hard dissonance. Yet further ratios such as 36/35 just get 'harder.' Thus the group of consonant or near consonant ratios is not very large at all. By contrast, the group of dissonant ratios is vast, seemingly beyond comprehension. Consonance is a small island in a vast sea. Consequently, consonant ratios form reference points or norms through which we attempt to navigate complexity.

The ancient poets and philosophers had various metaphors for the Spectrum of simplicity-complexity, or consonance-dissonance. They generally treated the Spectrum as opposing Forces or tendencies—called Love and Strife, or Peace and War. Strife is that tendency for simplicity to become complexity. It is the force that emanates the Many from the One, the force that differentiates forms and proliferates variety. On the other side, Love describes that tendency for complexity to resolve itself into simplicity—for the Many to revert to the One, for forms to merge. According to the old musical writers, Love and Strife are always found together in some sort of balance. Different tunings put
the balance in different places, but a good workable scale always has aspects of both. An overbalance of Love leads to euphony (not enough there to hold our interest). An overbalance of Strife can lead to incoherence (it no longer 'hangs together'). In the Greek mythology, Ares (god of Strife) married Aphrodite (goddess of Love). They together had a daughter appropriately named Harmonia.

IRRATIONAL RATIOS

The consonance status of a ratio is easily discernible in the numbers involved, but the metrical property proves otherwise. Given a ratio such as 5/3 or 16/15 one cannot easily determine how wide is the spread in pitch between the lower and upper tones. We can compute that 16/15 equals about 1.066 while 5/3 equals 1.666, so that the 5/3 is wider than the 16/15. But this calculation does not tell us anything about the character of the interval. During the 19th century scientists decided to standardize a logarithmic scale called Cents ('c' for short) that makes this issue clear. A 12-et semitone was defined as 100c, an octave thus 1200c. This measurement is quite useful for comparing the sizes of ratios to each other. For example, we see that 16/15 (or 111.731c) is somewhat wider than the 12-et semitone, while the ratio 18/17 (at 98.955c) is much closer to the 12-et version, this time slightly narrower. The major sixth 5/3 (884.359c) is considerably different from the 12-et version (900c). Every ratio has its own metrical characteristics.

The usefulness of the logarithmic translation goes beyond the clarification of size. It allows us to model the harmony in modular parts like identical bricks. Thus, using 12-et as our sample temperament, we can say that the minor third interval equals three steps or semitones in the system, whereas the major third interval equals four steps. Moreover, it allows us to conveniently add musical intervals. When we say that a minor third plus a major third makes a musical fifth, we are saying that three steps plus four steps make seven steps in the system. These additive operations prove to be much less cumbersome than the alternative of multiplying ratios. In ratios, 6/5 times 5/4 equals 3/2. In short, the logarithmic perspective proves to be both extremely useful and convenient.

However, it also has its price. The ratio 5/3 has a definite character or ethos, a perceived quality that is considerably compromised when tempered so sharp. Consonant ratios act as powerful norms or reference points within the pitch continuum. We tend to hear the more complex and dissonant interval as a 'twisting' or tampering, but still a recognizable version of the major sixth interval. The dissonant interval can best be described as more active and bright, perhaps brittle. The 5/3 norm itself is more rested, or softer, or more 'colored.' Aristotle, the teacher of Aristoxenus, made the classic justification for tempering when he said that a ratio can be 'relaxed' up to a point and yet still maintain its identity. Norms can be manipulated but not ignored.

One of the seminal foundational pillars of Harmonics is the existence of two types of ratios—rational and irrational. Rational ratios such as 5/3 have always been called arithmetical, for obvious reasons. Irrational ratios are called geometrical, since they initially appear in geometrical constructions. The classical Greeks had no use of logarithms, which make irrationals easy to explore. Their only avenue to this world came
through the making of an irrational division of a line segment using a geometrical figure in the construction. Irrational ratios have this connection with visual or spacial constructions, and yet they also inhabit the vibratory realm.

When we combine the infinite set of rational numbers with the infinite set of irrational numbers, we define the full set of Real numbers. They are properly conceptualized as a continuum or a plenum in the ‘number line,’ an analog perspective of division rather than the digital arithmetical perspective. The early Greek poets and philosophers described it as ‘the All,’ which is ‘full of what is.’

Irrational ratios have an interesting feature. They tend to be surrounded by rationals, but none of them quite matches the irrational position. They are like void points among the arithmetical. Take, as an example, the case of the square root of two—perhaps the most famous of irrational ratios. It divides the octave into a sonic half (600c) and is called a tritone or augmented fourth. The consonance 7/5 (at 582.512c) sits at least in the neighborhood. But if we want something closer than that we must choose between various dissonant alternatives, such as 45/32 (590.224c), or 38/27 (591.648c) or maybe 24/17 (597.0c). One can always find another ratio that’s a little closer, or even very close, e.g. 140/99 at 599.912c, but there is always some discrepancy.

One can safely say that all irrationals are dissonant in some way, yet they can also very closely simulate consonant rational ratios. We can see a familiar example in 12-et, where the irrational fifths (700c) sits quite close to the just norm 3/2 (701.95c). Thus the relation between rational and irrational ratios is communal—they inhabit the space of each other and coexist on the pitch continuum.

The proper place and status of irrational ratios has always been consistently controversial since ancient times. Aristoxenus favored an irrational system (72-et) in a tradition that flowed through the atomist Democritus and ultimately the Milesian philosophers. The gateway to the exploration comes through the geometrical construction of the Geometric Mean, whose theorem was attributed to Thales, the first Milesian and the father of Greek mathematics. Euclid, in his collection of theorems, demonstrated the construction that allows the irrational division of a line segment, but he also made it clear that he was totally opposed to the use of irrationals in musical tuning. His polemical stance was anti-temperament. All later writers lined up into two camps—for and against Aristoxenus, with the majority against. By late antiquity temperament was as good as buried, not to resurface again until the European renaissance.

Throughout history the opponents of temperament have presented several arguments. Some say that irrationals are simply illusory. They do not exist because they cannot be found in the Harmonic Series. However, we can plot their positions on a monochord. These critics just wish that they do not exist, but they are deluded. Some admit that irrationals exist but give them a slippery character, deficient in some manner, or expressing a special sort of dissonance not found in arithmetical ratios. However, this argument has no basis—one cannot detect an experiential difference between rational and irrational dissonances. They sound like each other. All the critics stress the fact that
irrationals are absent from the Harmonic Series and assume that musical ratios must come from the Harmonic Series. But we should question this assumption. As we will see, even arithmetical ratios can come from an alternative source. Moreover, rationals and irrationals need not be bitter rivals, since certain successful tuning systems exist that use both together.

The defenders of geometrical ratios also have their own arguments against the arithmetical stand. However, I refuse to take a polemical stand. Instead, in this essay, I make two arguments: The rivalry between rationals and irrationals is a red herring, since either sort can well simulate the other sort. Both inhabit the same psychoacoustical space. My second argument contends that the choice is more a matter of aesthetic intent than absolute value judgement. We choose a system according to its particular characteristics and how well these characteristics suit our needs.

RATIO SEQUENCES

Ratios are to harmony what colors are to painting. They define the most basic building tools. By mixing or combining them we can construct edifices such as chords and scales. These higher order relations of relations form aggregates of ratios in a sequence. In order to make the investigation of these sequences more convenient, I have presented ratios in a different manner. Rather than using the expression a/b and its reciprocal b/a we employ a:b. Here, in this rather more abstract expression, we must always specify the two pitch directions—upward and downward. Sequences can always be seen from two different perspectives, whose derivation is our fundamental principle of physics (frequency versus wavelength).

At this juncture the specification of pitches also becomes relevant. I have arbitrarily (for reasons of convenience) chosen pitch C as my reference 1/1. The choice has nothing to do with ancient practice, whether late Roman (where A is the reference) or early Greek (where letters were not used until Hellenistic times). Essentially, one must choose some pitch to stand for 1/1.

Here is a simple example of an arithmetical ratio sequence. Take 3:4:5:6. In its upward orientation it makes the pitches C F A C. Specifically, this means that we move from C up a 4/3 to F, then up a 5/4 to A, then up a 6/5 to C. This harmony is bounded by an octave, since 3:6 equals 1:2. An even easier manner to conceive the up sequence goes directly to the Harmonic Series. The numbers represent partials (or overtones) of some fundamental. The sequence then says that—given that C is the third harmonic, then F is the fourth harmonic, A is the fifth harmonic, and the octave C the sixth harmonic. The numbers refer to the Harmonic Series, in this case an assortment whose fundamental is on F. Numbers thus refer to frequency relations.

In the alternative downward perspective, the sequence 3:4:5:6 yields the pitches C G Eb C. Here the numbers refer to string lengths, to a monochord situation. A music wire of some length is stretched over a soundboard and well fixed by end-bridges. It is then divided into six segments. A moveable bridge is then placed at a position, for example
position 3—the middle of the string, called by the Greeks the mese. The sequence now means that we move from C down a 3/4 to G, then down a 4/5 to Eb, then down a 5/6 to C. Seeing it the monochord way as discrete positions on an arithmetical division proves the easiest way to interpret the downward perspective.

Either avenue yields a harmony that is an architec tonic whole. It makes a gestalt in which parts are also wholes. The sequence 3 4 5 6 not only embodies the ratios 3/4, 4/5, and 5/6, but also 3/5, 4/6=2/3 and 3/6=1/2. The relations are best tabulated as an interval triangle of relations. Within our sequence we see a whole collection of disparate ratios housed within an octave, but they nevertheless define a single whole, a single harmony that embodies and integrates these subset harmonies. Many such arithmetical harmonies are possible, each defined by its own ratio sequence.

TWO TUNING METHODS

There have always been two methods of setting arithmetical ratios on strings. I call them the harmonics method and the monochord method. First the harmonics method.

It involves a comparison between the harmonics of strings. For example, say that we want to tune our G string a 3/2 musical fifth above our reference string C. Compare the third harmonic of the C string to the second harmonic of the G string. They should form a beatless unison. If they do, then the interval between the C and G string passes the test. This method hangs on the ability to produce clear harmonics and to judge a beatless unison. Unfortunately, the harmonics become increasingly difficult to sound as we move up the series. It's easy enough to set a 3/2 or a 5/3, but try a 11/9—not easy. The major limitation of this tuning method is this restriction to simple ratios.

One way to get around this restriction is to employ an indirect method. Say that we want to set the pitch B a 15/8 ratio above our reference C. The direct method just isn't workable. The indirect method uses an intermediate string. In this case tune the G string a 3/2 above C, then tune a 5/4 from the G string to give B. It works because 3/2 times 5/4 equals 15/8. Although it's not possible to tune the B string directly from C, we got it using the G string as a stepping stone. Here I only needed one intermediary, but one could also use several. But at this point another limitation of this method comes into play. Tuning errors are accumulative. After a series of such operations the goal is no longer very accurate.

Many ratios cannot be set at all by the harmonics method, either by direct or indirect means. Take the ratio 17/13, an F semi-flat. Perhaps by using a very long string and a calibrating device one could just locate the 17th harmonic, but it would be so weak that it is hardly useable. Thus the harmonics method has real restrictions.

Such restrictions do not exist in the other tuning method, the monochord method. One is free to incorporate any numbers that we want into our string divisions. Moreover, we need only one string and can eliminate intermediaries. For its universal accessibility monochords prove to be the royal route to the exploration of arithmetical ratios. All the
available evidence confirms that the ancient tuners neglected the harmonics method and use the monochord method exclusively.

However, the monochord method also has its in-built deficiencies. In order to make a sound the moveable bridge must deflect the string a little, altering the tension on the string, and making a discrepancy between the theoretically correct position and the desired musical interval. This error (the ‘fatal flaw’ of monochords) can be reduced by using a very long string length, high tension, and a minimal deflection on the bridge, but it cannot be eliminated altogether. Ancient tuning experts like the scientist Ptolemy were quite aware of the problem. They recognized that monochord expertise is an art as much as a science. The calculated position must always be ‘tweaked’ by ear for the desired result. This situation led to the ancient notion of theorectica and practica, of logos versus perception. It led to the notion of an ideal mathematical world being approximated in the practical world.

ASYMMETRY

Ratio sequences can be classed as either symmetrical or not symmetrical. Symmetrical harmonies make the same pattern of pitches both in the upward and downward orientation. Asymmetrical harmonies do not. Later I will address the issue of symmetry, its types and characteristics, but here I focus on asymmetry. The vast majority of harmonies prove to be asymmetrical, including our sample sequence 3:4:5:6. Such sequences make a different pattern in the two orientations.

Asymmetrical sequences always have this ‘yin-yang’ pairing. They prove to be reciprocals or flip-overs of each other. In the case of the sequence 3:4:5:6, the upward pattern makes a major triad (F major) while the downward pattern makes a minor triad (C minor). These two types of triads are asymmetrical and related to each other.

We can pick symmetrical sequences out of the materials of the Harmonic Series, but the series as a whole is asymmetrical. This fact constitutes one of the fundamental pillars of the science of Harmonics. It forces us to posit the notion of its reciprocal—called the Sub-harmonic Series. The first six harmonics of the Harmonic Series (on C) yields a major triad (C major), while the first six members of the Sub-harmonic Series yield a minor triad (note, F minor, not C minor). We can express this relation by using the sequence 4:5:6:8. The upward Harmonic Series orientation gives C E G C, the C major triad. The downward orientation gives C Ab F C, the F minor triad. More complex sequences make for similarly complex reciprocals. The Harmonic Series and its reciprocal seem to need each other or compliment each other. Together they form the ratio possibilities that fill up harmonic space.

The complimentary nature of the Harmonic Series and its reciprocal surfaces in various contexts. We have just seen it in the last section on tuning methods. The Harmonics method comes from the Harmonic Series, the monochord method from the Sub-harmonic Series. It will come up again later in consideration of the Musical Means and the Musical Genres.
The notion of the Sub-harmonic Series has generated enormous historical controversy. What status does it have? Contenders in the ongoing polemic fall into two broad camps that I call reciprocity theorists and turbidity theorists. Both camps include both moderates and extremists.

Reciprocity theorists argue (rightly) that the mathematics is clear and unambiguous. We should therefore accept its implications and afford the minor triad equal status with the major triad. The more moderate reciprocity theorists concede the arguments of the turbidity theorists and admit that the minor triad is less consonant than the major triad. However, they argue that it is expedient or useful to maintain the reciprocity framework in the interest of building an elegant and consistent musical theory. I have sometimes used this approach myself in several of my papers (e.g. On Chord Progressions).

The turbidity theorists downplay the reciprocal relation between the minor and major triad. Instead, they see the minor triad as a more ‘clouded’ or turbid version of the major triad. They argue (rightly) that the minor triad embodies less consonance than the major triad. We can prove it through the examination of the inherent Difference Tones and other matters. Hence the structural framework that reciprocity affords is largely delusional and should be abandoned. Build a new theory of music on a different footing. The more moderate turbidity theorists will still allow some aspects of reciprocity into the fabric, coming closer to the moderate reciprocity theorists.

The more extreme turbidity theorists want to sidestep or eliminate Sub-harmonics altogether and derive everything from the Harmonic Series alone. They argue (rightly) that Sub-harmonics do not exist in nature—everything vibrates in partials of a whole. Sub-harmonics exist only under special ‘laboratory’ conditions. Hence we should abandon reciprocity altogether. As an example of this way of thinking, take the ratio 4/3. Some will argue that it has inferior status (or no status at all!) because it isn’t found in the overtones. Instead of using this ratio of 4/3 (498.045c) we should rather use 43/32 (511.518c) because it sits in the Harmonic Series. Some will argue that the 4/3 is impossible, even though it has been used for thousands of years in practical tuning.

In my opinion the extremists go too far in denying Sub-harmonics. After all, the ancient Greeks, having a monochord orientation, derived their arithmetical ratios from the materials of the Sub-harmonic Series, not the Harmonic Series. As I will explain later, the Harmonic Series and its reciprocal also interpenetrate each other in subtle ways. Hence it seems unnecessarily extreme to deny one side of the reciprocal pair altogether.

Reciprocity theorists constitute the historical mainstream. Perhaps this is because most researchers into Harmonics have been mathematicians rather than musicians. Such people tend to be hardwired into finding and valuing symmetries and elegant formulas. Turbidity theory has come into its heyday since the 19th century acoustical research of Helmholtz and others. It aims to put more emphasis on psychoacoustical realities, less on
a priori theoretical models. The aim is commendable, but some balance must be found to accommodate both sides of the issue.

At any rate, even if we downplay the importance of reciprocity regarding the Harmonic Series, it still dominates the field in other contexts: the complimentary nature of frequency and wavelength, the complimentary directions of pitch, the two complimentary tuning methods, the complimentary pairing of harmonic structures, and (we will see) the complimentary Musical Means. Reciprocity permeates the whole territory, so I would caution an abandonment of the Sub-harmonic Series. I prefer to treat it as the ‘mirror world’ or the hidden world that we can make manifest through a monochord.

PRIME LIMITS

We come to another fundamental pillar of the science, this one perhaps the greatest of all rocks. Call it the Law of Primes. It places numbers into families that have kinship relations.

Integers can be sorted into two groups—prime and composite. A prime number has no factors other than itself and one. The first prime number is 2, the only even prime number. The prime number series begins 2, 3, 5, 7, 11, 13 and so forth in a pattern still not entirely understood. Whether the number one should also be deemed prime is an ongoing argument, but the consensus says that a prime must be an integer greater than one.

The class of prime numbers is infinite, as proved by Euclid. Numbers that are not prime can be factored into a product of primes in one way only. Euclid also proved this, and called it ‘fundamental theorem of arithmetic.’ Indeed, it’s the ground of Harmonics. The first composite number is 4, which breaks down into 2 times 2 or the square of 2. Every number can be expressed as a unique product of primes, putting it into a family of numbers that share the same roots. Composite numbers ‘marry together’ prime factors. They are like the children, or tribes, or descendants of primes. Primes are the parents, the ancestors, the source. Thus prime numbers define the underlying roots, elements or powers that support the vibratory realm.

Remember that the tuner’s problem is to maximize the consonant ratios in the scale, while keeping complexity at bay. These aims are mostly accomplished by restricting the variety of primes allowed into the ratios. Nevertheless, one can hold differing opinions over the level of restriction, over which primes are permitted in the harmony. These differing opinions lead to conflicting schools of musical tuning. Here is a brief introduction to the families of harmony. We begin with maximal restriction and gradually permit more inclusion.

2-Limit harmony (2-L harmony for short) permits only multiples of the prime number 2. All higher primes are excluded. We are then left only with ratios such as 2/1,
4/1, 8/1, 16/1 and so on. They form only 'empty' octaves. Consequently, this level of restriction is excessive.

3-L. Harmony permits only powers of the prime numbers 2 and 3. Now we can make an infinite profusion of ratios, since no power of 3 matches any power of 2. This approach also as great utility, since one tunes by using only octaves, fifths and fourths. The system also allows one to conveniently model tuning as a line of fifths-fourths. All of these advantages have made 3-L harmony extremely widespread (e.g. in faraway China) and also extremely old and long-lived (e.g. in old Babylonia). Among the Greeks who championed this approach we find Pythagoras, Philolaus, Euclid and many other writers.

However, 3-L harmony also has a difficult side. Ratio numbers quickly become enormous. Note how the numbers balloon in our line of fifths: 1/1, 3/2, 9/8, 27/16, 81/64, 243/128, 729/512, 2187/2048 and so on. Maintenance of strict 3-L harmony has always been a bit of a tour-de-force, since one must deal with large numbered monochord divisions. The semitone ratio 2187/2048 (113.685c) can be closely simulated by the 5-L ratio 16/15 (111.731c). Such observations promote a relaxation of restrictions to give 5-L harmony. Here one is allowed powers of the primes 2, 3, and 5.

5-L harmony has definite attractions. It allows the use of beautiful consonances like 5/4 and 5/3. It tames the numbers so that very practical scales can be made without enormous divisions on the monochord. It provides a large and pleasing variety of useable scales. Moreover, it still retains 3-L ratios as a significant subset, so that one can continue to use the desirable 3-L material. For all of these reasons 5-L harmony constitutes the historical mainstream of tuning tradition in India and the Middle-East. In Greece various writers offered 5-L tunings, notably Didymus and Ptolemy.

Our sample sequence 3:4:5:6 is also a 5-L. harmony. We could convert it into a simple 3-L harmony by reducing it to 3:4:6. Here is an example of a beautiful (and widely popular) 5-L diatonic scale, which sits in the octave 36:72. The sequence goes 36:40:45:48:54:60:64:72, the upward pitches C D E F G A Bb C and the downward monochord pitches C Bb Ab G F Eb D C. First rate alternative 5-L diatonic scales also sit in the octave 30:60 and 24:48. Another group of alternatives that have secondary status can also be defined. The point here is that 5-L harmony offers such a wealth of useful patterns that we do not need to seek further.

Nevertheless, Archytas (a friend of Plato) recommended 7-L. harmony. Since the 7-L contains the 5-L material as a subset, its resources are truly vast. We can arguably justify its use in the interest of simplifying numbers. For example, the 5-L augmented second interval 75/64 (274.583c) becomes 7/6 (266.871c). 7-L harmony forms a sort-of esoteric extension of the 5-L 'major-minor' norms. It makes some very elegant harmonies, but it also tolerates greater complexity.

Archytas is the earliest surviving writer that one can unambiguously call a progressive tuning theorist. The 3-L camp comprises the conservatives who perpetuate old traditional methods. Supporters of 5-L harmony make the mainstream, which one can
also class as conservative in that 5-L harmony is age-old and widespread. Archytas broke through a ceiling and opened a gate that allowed in yet higher prime numbers. Eratosthenes used a 19-L ratio, Didymus a 31-L ratio. But the greatest of the progressive classical theorists, the scientist Ptolemy, specified several 7-L harmonies (possibly in homage to Archytas), some 11-L and even a virtuoso 23-L harmony. In this highly progressive tuning philosophy one eliminates any restriction on the primes. Ptolemy permits any prime as long as it suits the aesthetic intent. The rationale for using an exotic prime is the same as above—in order to simplify the numbers in the monochord sequence.

Thus we can see several camps among the ancient theorists: conservative, mainstream, and progressive. The scientist Aristoxenus, who lived about a half century after Archytas, was undoubtedly influenced by Archytas in his progressivism. But he took a yet more radical route. He abandoned arithmetical ratios altogether in favor of geometrical ratios.

THE MATRIX AND OTHER MAPS

Aristoxenus wrote about all aspects of his music culture, including its history and its controversies. He mentions a group of tuners that he calls ‘the Harmonists,’ who make ‘close-packed diagrams’ (triangular-hexagonal arrays). He may be referring to some Pythagoreans, since another garbled tradition has tuners making ‘pebble patterns’ or diagrams formed by pebbles. These stories support an ancient awareness of the Tri-axial Matrix of tuning operations. Setting historical issues aside, the matrix is natural to the tuning environment, since structures are defined by relations (logos).

The matrix is a valuable tool for understanding harmonic structures, since every harmonic pattern has its own signature in the pebble pattern. Moreover, it is independent of the two tuning methods, so that we can look at both the harmonics and sub-harmonics. No matter which method we use to tune from C to G, the relation is expressed by an unambiguous address on the matrix. Here’s how it works.

2-L harmony places a special pebble, the ‘mother bead,’ to signify the source, our pitch reference C. Since harmonies repeat at the octave, one octave represents any octave. Thus we need only a single pebble for 2-L harmony.

3-L harmony can then be mapped as a horizontal line of pebbles, which define the various modes of 3-L harmony. 5-L harmony expands the line into a surface in which a major triad makes an up-pointing triangle, a minor triad a down-pointing triangle. Using this address system one can then define any and every type of 5-L harmony. This grid expands toward infinity in every direction, so that it eventually runs into large numbers, but the best harmonics stay close to the source. Our matrix expresses a progression of dimensions—point (2-L), line (3-L), and surface (5-L). The next member, 7-L, makes a solid, but here the utility of the model starts to break down. It is possible to use substitutions so that desirable 7-L ratios can be mapped within the 5-L array. Thus 7-L harmony can be at least accommodated, but higher primes like the 11-L make another
story. They become more difficult. Thus the Tri-axial Matrix has its own limitations and is only ideal for 3-L and 5-L harmony. But then, this is the mainstream anyway, where most useful or practical harmonies are found. Consequently the matrix remains a valuable tool.

The matrix is not the only ancient method for mapping arithmetical ratios. Another is called the Lambdoma. It consists essentially of a 90 degree ‘x and y’ axis chart. Along one axis sits the ‘over’ ratios of the Harmonic Series 1/1, 2/1, 3/1, 4/1 and so on. Along the other axis sits the ‘under’ ratios 1/1, 1/2, 1/3, 1/4 and so on. Every possible ratio can then be tabulated between these two horizons. The border between them sits on the diagonal line that defines 1/1, 2/2, 3/3, 4/4 and so on. In effect, the Lambdoma consists of a grand interval triangle for the combined Harmonic and Sub-harmonic Series. Unfortunately, apart from defining the ratios themselves, it gives us very little additional information. Unlike the matrix, where significant or useful harmonies make significant pebble patterns, the Lambdoma tells us practically nothing useful.

A variant of the Lambdoma approach was also used to generate the 3-L line of fifths. Here the ‘x and y’ axis refers to the powers of 2 and 3. Nichomachus and other supporters of 3-L harmony employed it to enhance the mystique of their preferred tuning, but it gives no new information not already found in the matrix. The essential part is the line of fifths that define 3-L tuning.

A fourth model for mapping ratios is called the Circular Graph. Since patterns repeat at the octave, we define the octave as a circle of 360 degrees. Every other ratio can then be mapped as degrees of arc by using a logarithmic conversion. As a result every pattern of harmony makes a unique geometrical pattern on the circular graph. This graph is quite useful since it makes explicit and visual all aspects of symmetry. The recognition that a harmony is symmetrical is not obvious while examining the ratio sequence or the field matrix. Hence the usefulness. Some writers contend that ancient tuners knew the circular model, due to an obscure reference in Plato to ‘bending the harmony into a circle.’ However, I doubt that they possessed it, since the approach is entirely dependent on logarithms. But they could conceivably have made rough diagrams, showing a grasp of the concept.

NUMEROUSITY

In our sequence 3:4:5:6 the octave 3:6 is identical in meaning to 1:2. The numerosity has only been increased so that it can act as a least common multiple that will accommodate larger numbered ratios like 3:4 and 3:5. It allows these further ratios to be admitted into the set or family. The octave acts as a sort-of ‘house’ or space (the classical metaphor is the mixing bowl) where ratio diversity is generated. The number 2 is traditionally or mythologically female, the goddess of the matrix, the womb of harmony, the spinner of the vortex of complexity. The number 2 introduces a microcosm into the macrocosm of the pitch field. We need only the scale for one octave in order to work out the scale for any octave. The octave has this ‘floating’ property that permits the single
mother bead on the pebble pattern. It has the power to take on endless different numerosities.

As an example of the 'power spinning about the double,' as Plato put it, say we want to include the 6:7 ratio in our sequence. One way to place it simply doubles 3:6 to 6:12, so that we can have the 7-L sequence 6:7:8:10:12. Our sequence has now become a subset within a more complex harmony. Higher numerosities potentially reveal wider resolutions of the harmonic field. Harmonies are perfectly suited to the concept of sets, subsets, and supersets. Here our 7-L sequence ruled by 6:12 forms a superset of the 5-L sequence ruled by 3:6. You may not have noticed it, but 3:4:5:6 also forms a subset of the prominent 5-L diatonic harmony ruled by 36:72 given earlier. There it 'hides' as the sequence 36:48:60:72. Harmonies thus nest within each other.

Our sequence 3:6 is simple enough that we call it a musical chord. The 36:72 division has accumulated more pebbles in the pattern. It has become a full-blown musical scale. Being 5-L, the majority of the intervals will be consonant. For example, 36.45=4.5 and 40.60=2.3. But within its copious interval triangle it has also picked up some dissonances, e.g. 40.45=8.9 and 45.48=15.16. Significantly, it has also picked up a dissonance that cannot be further reduced, here 45:64—a tritone interval (609.777c). In some sense this ratio is the ruler or cause of the division 36:72. We had to choose this numerosity in order to allow the tritone into the scale. Every division has at least one special dissonance that cannot be reduced into smaller numbers.

Most musical scales have five to eight notes. Beyond this point we tend to model harmonies as collections of scales within a wider framework—a pool of possibilities. A significant 5-L chromatic (12 note) set sits in the division 360:720. But this pool is still relatively cramped. Traditional tuning cultures went much further, witness India's twenty-two notes or the medieval Arabic seventeen. Out of this cornucopia of possibilities many different scales can be defined.

Numerosity has no bounds and the pebbles can accumulate into an intricate pattern on the matrix. But eventually we must come face to face with the hard issue. Where do we draw the boundary, the limit? What is the maximal permitted numerosity? Preferably it should be some grand number that acts as a superset to every significant harmony. But what is it? One can only give steps along the way, such as the 5-L harmonies 1800:3600, or 4320:8640 or 12960:25920.

The ancient scientists tended to conflate large musical numbers with calendrical numbers and mythological numbers. One of the great pre-occupations of ancient philosophy-science was the discovery of the grand cycle that integrates all of the lesser cycles. It is generally described as the Great Year and focuses on the temporal dance of the planets. Eventually, after so many years, all of the planets must come back into a grand conjunction. The problem is inherently musical, so it is no surprise that certain valuable numbers from the musical perspective infiltrate astronomical contexts. The candidate for a Great Year is generally also an interesting monochord division. The large mythological numbers of Babylonia, India and Egypt are all, with very few exceptions,
numbers that use multiples of 2, 3, and 5. However, we should never forget that practical music makers never had a need for such issues—it was the domain of the theoretical mathematicians. Players had more than adequate resources in a diatonic division like 24:48 or 30:60.

No matter what route we take, 3-L, 5-L or what I will call \textit{n-Limit harmony} where no restrictions remain, numerosity can grow toward infinity. In this situation the harmonic field becomes saturated and the monochord division approaches the continuum, the plenum of infinite divisibility. Here in the Unlimited the classical force of Strife has its zenith. But complexity leads to unworkability. The tuner must re-impose the force of Love by establishing Limits.

THE MUSICAL MEANS

Archytas is the first surviving writer to explain the three classical Means (arithmetic, harmonic, and geometric). However, it is unlikely that he discovered them himself. The Geometric Mean (GM) was universally attributed to the Mileian Thales. The Arithmetic Mean (AM) and the Harmonic Mean (HM) are most likely yet older, especially the AM. Later, Eudoxus expanded the set to six Means, exhausting all the possible ratios consisting of combinations of the mean and the two extremes. During Hellenistic times Eratosthenes, Pappus and Nichomachus recognized ten Means. But these extensions have little relevance to musical tuning. The three classical Means suffice for everything.

The GM is a special case because it alone generates irrational ratios. We will examine it later. The AM and HM form a tight reciprocal pair that are together called the \textit{Musical Means}. With them the world of arithmetical ratios opens up.

The oldest, simplest and most ‘primitive’ Mean, the AM, is also the most practical and useful. It can always be demonstrated on a monochord. The mathematicians established formulas for deriving ‘b’ as a mean between the extremes ‘a’ and ‘c.’ In the case of the AM the formula goes \( a:b = b:c \). The solution to this formula defines b as \( a+c \) divided by 2. The use of a formula reflects the great importance of ‘taking the mean’ or ‘taking the middle’ in monochord procedures. But the operation is so simple that the formula itself is not even needed. For example, find the AM of the octave ratio 1:2. Just double the numbers and insert the middle term to give 2:3:4 (C F C). Such monochord operations define aspects of the Sub-harmonic Series. The AM of the fifth 2:3 (C F) yields 4:5:6 (C Ab F). A ratio like the 5-L major sixth 3:5 (C Eb) is even easier to handle—it yields 3:4:5 (C G Eb). More difficult ratios use the identical treatment. The arithmetic is only a bit more tedious. For example, the 3-L major sixth interval 16:27 (C \( \flat \text{Eb} \)) becomes 32:43:54 (C G+\( \flat \text{Eb} \)). One can easily apply such a procedure to any interval and mark it on the monochord.

Note that when we take the AM, the solution does not respect the family of the generator. Spectacularly, the 3-L ratio 16:27 gives ratios that are 43-L. If we want to maintain the 3-L character in our division, we should replace 32:43:54 with 32:48:54.
which equals 16:24:27 (C G ∩Eb). But then we no longer have the exact mean between 16:27, rather a different division. This tendency for the AM to produce higher primes makes the procedure the darling of the progressive tuners. Conservative tuners were more reticent and restricted its use, but it's hard to avoid such a simple process.

There's nothing esoteric or difficult about using the AM. It amounts to placing a fret position halfway between two existing fret positions on the monochord. It's so basic but it goes so far. We see it being used by the progressive tuner Didymus when he wanted to generate two quarter-tones from the 5-L semitone 15:16 (C B). His solution was 30:13:32 (C B+ B). The operation divides the semitone roughly in half, though it's never an exact half. In this case 30:31 equals 56.767c and 31:32 gives 54.964c. No doubt Didymus accepted this high prime number in order to keep the overall monochord division numbers lower.

In another example with similar aims, take this instance in the progressive tuner Eratosthenes. He wanted to divide the 5-L whole tone 9:10 (C Bb) into semitones. The solution is 18:19:20 (C B+ Bb). Here the semitones are mid-sized. 18:19 equals 93.603c and 19:20 equals 88.801c. Again, he probably accepted the high prime 19 in order to make an elegant overall division. If he had insisted on a strict 5-L solution he would have had to jettison the AM and go for a sequence with larger numbers—45:48:50 (C B Bb). Here the semitones are no longer roughly equal but rather quite variant in size. 45:48=15:16 (111.731c) and 48:50=24:25 (70.673c).

Ptolemy sometimes used a variation of the procedure that we can call a 'double AM.' Here is an example. We want semitones derived from the 3-L whole tone 8:9 (C \Bb). Instead of using the standard method to generate 16:17:18 (C B+ \Bb) he multiplies the ratios by three to give 24:25:26:27 (C B B- \Bb). Then we jettison one element (say the 13-L element 26) to end up with the 5-L sequence 24:25:27 (C B \Bb). The aim here is to generate two semitones in which one is roughly double the size of the other. 24:25 equals 70.673c, while 25:27 equals 133.238c. This method is a refinement of the traditional procedure, and could theoretically be further extended.

The HM is the reciprocal cousin to the AM. While the AM refers to the monochord and Sub-harmonics, the HM refers to the Harmonic Series and frequency relations. For example, find the HM of the octave 1:2. Again, we double the numbers and insert the middle term to make 2:3:4 (C G C). The numbers refer to the material of the rising scale of the Harmonic Series. In referring to the Harmonic Series it suffers from the same limitations that we saw earlier—large numbers are just not feasible on the music wire. Nevertheless, it's theoretically very easy to find the AM and HM of an interval. Staying with 2:3:4, the first (the larger) ratio 2:3 (G) expresses the HM and the second ratio 3:4 (F) the AM. This method works for any interval.

However, the situation becomes a little more complicated because the ancients ignored (or didn’t see) the Harmonic Series. They wanted to place the HM on their monochords. For this they needed another formula, duly provided by the mathematicians. In the HM a-b divided by a equals b-c divided by c. The solution then gives b equal to
2ac divided by a+c. With this formula we can place the desired ratio on the proper place on the monochord. So for the ancients the HM was a bit more complicated or indirect than the AM. Consequently it didn’t have quite the status of the AM. Its secondary status was reflected in the traditional name ‘sub-contrary.’ According to the record Archytas changed the name of this mean from the Sub-contrary Mean to the Harmonic Mean. No doubt Archytas, a musical genius, recognized the reciprocal nature of the two Musical Means, but for the ancient tuners the monochord perspective always comes first.

THE PROGRESSION OF MEANS

Scattered throughout the ancient philosophical literature are various passages about musical tuning and its theoretical foundation. Clearly the philosophers had an active interest in the subject. Perhaps the single most spectacular passage I have ever found occurs in the Epimomis, a post-humous dialogue either by Plato or one of his pupils. The author recommends that we examine the Musical Means of the ratios 2/1, 3/2, and 4/3. He connects the construction with the progression point, line, surface and solid—thereby hinting at the matrix. When we follow his directions, it lays bare the theoretical basis for the Musical Genera (diatonic, chromatic, enharmonic), and the important concept of resolution. It’s quite a revelation.

The ratios 2/1, 3/2, 4/3 mark the start of the Epimoric Ratio Series. Epimoric ratios have the form n+1/n. Classical theorists like Archytas and Ptolemy greatly valued epimoric ratios and tried to maximize their use. In the following construction every ratio, including the by-products, form epimores. These ratios define important pillars within the deep structure of harmony.

The progression of means begins with the octave 2/1 (the point). Extracting the Musical Means yields 3/2 and 4/3. These two ratios relate to each other by the 3-L wholetone 9/8 (203.910c). The musical fifth and fourth clearly supports the establishment of 3-Liimit Harmony, the line. The construction also founds the Diatonic Genus that has a natural home in the fabric of 3-L harmony. The Diatonic Genus refers to a resolution of harmony into patterns of wholetones and large semitones. One can also make such patterns in 5-L harmony, but in 3-L harmony one can make only this sort of pattern—unless one institutes an inordinately long line of fifths. For this reason the Diatonic Genus has a natural home in 3-L harmony.

The Diatonic Genus can also be implemented by an irrational division that resolves the octave into large, sonically equal-sized semitones. In this context the Diatonic Genus refers to 12-et with its step semitone of 100c. The system 12-et is best seen as a modification (tempering) of the line of fifths. In both the temperament and 3-L tuning one is committed to the horizontal axis of the matrix.

We proceed to the next epimore, 3/2. It yields the Musical Means 5/4 and 6/5, which relate to each other by the small or chromatic semitone 25/24 (70.673c). We have now given birth to the basis of 5-Limit harmony and the three axes of the tri-axial matrix. This construction also supports the resolution of the octave into small semitones, called
the *Chromatic Genus*. Such harmonies use small semitones as the smallest element in
their scales. Archytas used the 7-L version 28/27 (62.961c). Again a significant tempered
analog can be defined for the Chromatic Genus resolution. It is 19-et, a system whose
step interval is 63.158c. As a tuning approach 19-et essentially aims to tune the 6/5 minor
third practically pure and accept what happens with the rest. Setting aside the tempered
system, Chromatic Genus harmony orients itself to the 6/5 axis of the tri-axial matrix.
Indeed, we can model 19-et as a modification of the minor third axis of the matrix.

Following our order of epimores, the next ratio is 4/3. It gives the Musical Means
7/6 and 8/7, which relate to each other by the 7-L minimal enharmonic 49/48 (35.697c).
We have now generated the `solid’ of 7-Limit harmony as well as the Enharmonic Genus.
Such harmony uses as its smallest component an enharmonic, also variously called a
diesis or generally a quarter-tone. Of course, this small interval can be various sizes.
Archytas used the 7-L ratio 36/35 (48.770c). Ptolemy used one as small as the 23-L ratio
46/45 (38.051c). Again, a significant tempered system supports the resolution of the
octave into enharmonics. It is 31-et, whose step interval is 38.710c. This system shows a
bias toward the 5/4 axis of the matrix. It recommends that we tune the major thirds
practically pure and accept what happens with the rest. It can be modelled as a subtle
modification of the 5/4 axis. Enharmonic Genus harmonies tend to orient around the 5/4
axis of the matrix.

At this point Plato stops his progression of means. I can imagine several reasons
that he would do so. First of all, he has reached the solid dimension, so it’s difficult to go
on. Furthermore, he has also exhausted all three biases on the tri-axial matrix, so it seems
a fitting place to stop. Additionally, the next epimore (5/4) generates ‘9-L harmony,’ but
9-L harmony doesn’t exist. The number 9 has the distinction of being the first odd
number that is not a prime. We need to move on to the following epimore (6/5) in order
to achieve 11-L harmony. This lacuna in the sequence of primes makes a natural
boundary. Finally, one could choose to stop here because enharmonics or quarter-tones
form the smallest intervals to be accepted as independent elements in the scale. This
appears to be the case in music cultures all over the world throughout history. Even in
18th century Europe tuners argued over the enharmonic distinction between G# and Ab.
Some wanted it, others wanted rid of it. Ancient Greece followed the typical pattern by
welcoming quarter-tones but rejecting smaller intervals as impractical, subliminal, or
‘atomic.’

Nevertheless, it proves highly instructive to continue on with the progression of
means. The following epimore 5/4 generates the Musical Means 9/8 and 10/9, which
relate to each other by the Syntonic Comma 81/80 (21.506c). As stated above, no new
dimension is produced. Indeed, this resolution sits comfortably within 5-L harmony. But
we have now generated what can be called the *Commatic Genus* or commatic resolution
of the octave. Commatic harmony has the fascinating (some say exasperating) character
of presenting comma-shifted versions of the intervals. We can already see it in the means,
where the 5/4 major third (C E) is divided either into 9/8 (D) or 10/9 (D), alternative
comma-shifted whole tones. Anyone who works with 5-L harmony soon runs into
commas and comma-shifted intervals. The axis bias that we saw in the Diatonic.
Chromatic, and Enharmonic Genus has entirely disappeared to reveal the 5-L matrix itself in all its glory.

The commatic resolution of harmony can also be expressed in the significant irrational division of 53-et. Here one step equals 22.641c. This system provides a valuable perspective on harmony, since the intervals of the matrix can be assigned a size in comma numbers. For example, an enharmonic makes two commas, a chromatic semitone three commas, a large diatonic semitone five. The 10/9 wholetone makes eight commas and the 9/8 wholetone nine commas. Even setting aside the irrational division, the materials of 5-L harmony prove to be amenable to this treatment.

It appears that the comma was widely recognized very early on in history. The ancient Chinese mathematician King Fang, while computing a long line of musical fifth ratios, noticed that the fifty-fourth member came very close to the beginning. In other words, the octave consists of fifty-three commas. The oldest text from India that discusses musical tuning uses the comma (called a sruti) as the primary building block. In Greece, Philolaus observed that the 9/8 wholetone measures nine commas in size, implying that the octave comprises fifty-three commas. During the ‘80’s I used to maintain that Philolaus knew or sanctioned 53-et. But now I’m skeptical of this position. He may well have known the octave of fifty-three commas without any support of the irrational division. Instead, being a conservative, he (like the Chinese) would have put it into a 3-L framework, using the 3-L Ditonic Comma ratio 524288/531441 (23.466c). Philolaus is the earliest Greek writer to acknowledge the comma. However, it is highly unlikely, as some contend, that he discovered the comma.

Moving on, we come to the Musical Means of 6/5, yielding 11/10 and 12/11. They relate by the interval 121/120 (14.367c). Here we define the world of 11-Limit harmony and a double-clismaic resolution of the field. In this resolution a large group of neutral intervals appear that have no place in 5-L or 7-L harmony. The resolution is also represented by the irrational division 89-et, with its step interval at 13.483c.

Next comes the epimoric 7/6, yielding the Musical Means 13/12 and 14/13. Now we have produced 13-Limit harmony and a semi-commatic resolution, since the means relate by 169/168 (10.274c). It has a significant irrational analog in 118-et, whose step sits at 10.169c.

Finally we come to the ratio 8/7, the last consonant epimoric in this series. It makes as Musical Means the diatonic semitones 15/14 and 16/15, which relate to each other by the 7-L clisma interval 225/224 (7.712c). Since fifteen is not a prime, no new dimension is produced as this division sits comfortably within 7-L harmony. The clisma resolution can also be represented by 152-et (step interval 7.895c). It makes a fitting place to stop our progression of means.

In summary, the progression of means has generated a series of structurally significant resolutions of harmony that can be expressed both in rational and irrational
ratios. The resolution levels go: diatonic semitone, chromatic semitone, enharmonic comma, double-clisma, semi-comma, and clisma.

Of course, this method can be carried on as far as desired, and it continues to generate structurally significant microtones. Here are two highlights. The means of 11/10 yield the mid-sized semitones 21/20 and 22/21, which relate by 441/440 (3.930c), a semi-clismatic or double-schismatic resolution represented by 301-et (step 3.987c). Last but not least, the means of 15/14 give the quarter-tones 29/28 and 30/29, which relate by 841/840 (2.060c), a schisma resolution represented by 612-et (step 1.961c). I have found over years of examining harmonic structures that the schisma, clisma, comma, and diesis keep recurring in numerous contexts. They can all be found on the tri-axial matrix.

ANOTHER VISION OF THE GENERA

In this last section I have extended the classical resolutions of harmony (diatonic, chromatic, enharmonic) into an open-ended series of finer resolutions. It can be followed all the way to the subliminal—the point where pitch differences become difficult to perceive. Surely the schisma (around 600 to the octave) is a good candidate for this boundary, though some argue that it actually sits between 200 and 300 notes to the octave—between a clisma and a double-schisma. Irrespective of how far we take this path, the path itself must be important since it uncovers key structural microtones like the comma and clisma. This progression of resolutions also has an optimal analog among equal-tempered tuning systems, even though the progression itself uses only arithmetical ratios. The progression also uncovers the successive primes-driven dimensions of harmony—3-L, 5-L, 7-L and so on. This whole elegant architecture was derived from the Epimoric Ratio Series by taking Musical Means and extracting their relation to each other.

When I drew up this set of diagrams in the ‘80’s, I had not yet made this grand synthesis. Instead I took the three classical Genera in an isolated fashion, standing alone and apart from the larger progression. It proves interesting to see the way that I interpreted the Genera at that time. I present it in the Diagram showing that the progressive integration of higher primes is illustrated by the concept of the three Genera.

The diagram largely speaks for itself. I base the Genera rather narrowly on specific harmonic structures of the 5-L matrix. In the Diatonic Genus I lump together 3-L and 5-L harmony and connect them to the comma-shifted versions of the Diatonic Ogdoad scale. Thus for example the 3-L line gives the semitone between E F (four commas in size) while the 5-L harmony uses E F (five commas in size). In the Chromatic Genus I use substitutions in order to place 7-L ratios on the matrix. Here the ‘triple-strand’ pebble pattern seems appropriate since it embodies chromatic semitones (e.g. between C and C#). In order to get the Enharmonic Genus I further expanded the former triple-strand into a ‘quadruple-strand’ of pebbles. This way I can find an enharmonic interval such as G# Ab or B# C. Here again I use substitutions in order to introduce 11-L ratios over the 5-L matrix, a tricky problem.
Looking at it now, several inadequacies come to mind. First of all, the three Genera are tied to specific, arguably arbitrary structures rather than resolution levels of harmony. Of course the chosen structures do form significant harmonies that illustrate intervals of the three Genera, but many other harmonies can also be used to illustrate diatonic and chromatic semitones. Secondly, the generation of the dimensions proves a bit awkward, since both 3-L and 5-L are grouped with the Diatonic Genus. Beyond this point substitutions are used to achieve 7-L and 11-L harmony. The construction is too fixated on the 5-L matrix. Fourthly, I give absolutely no mention of the Musical Means or the concept of variable resolution, as if these issues are irrelevant to the Genera. Yet the Genera themselves illustrate variable resolution.

During the '80's I had a clear conception of the levels of resolution of harmony. But I confined this concept to equal-tempered divisions like 19-et or 22-et. For example, I would say that 34-et is a higher resolution than 31-et. It simply has more notes between the octave. However, no value judgement is implied. Every resolution must be individually assessed, just like ratios themselves. Each has its own character of 'field.' Both of these resolutions I label enharmonic. In my progression of means I have chosen 31-et over 34-et (and others) as the optimal representative of the enharmonic resolution because 31-et has an edge over 34-et, although both are good. 32-et and 33-et are just plain lousy. 31-et is superior because it has excellent simulations of 7-L ratios, a fitting characteristic for an enharmonic resolution. By contrast, 24-et (another decent exemplar) has poor 7-L ratios. All of the alternative divisions in the neighborhood must be investigated and appraised as forms of an enharmonic resolution.

I still use the concept of resolution in this manner but I've expanded its territory. It's also possible to make an enharmonic resolution harmony using only arithmetic ratios. The resolution concept began while investigating equal-tempered divisions but it need not be confined there. In the diagram I used some interesting harmonies but I could better have chosen a wider field that comes close to the 31-et field. Then it is an unequal division rather than an equal division of this wider field. The structures given in the diagram just form subsets of this wider field, that tends to orient itself around the 5/4 axis of the matrix.

The diagram also misses any mention of the Musical Means. Yet I was well aware of their relevance for monochord exploration. This usefulness is illustrated in the following diagram. There I generate various sizes of quarter-tones by using the two means. But I had not yet realized the implications of Plato's potent procedure. Perhaps Plato did not realize the implications himself—his treatment of musical issues is generally cryptic and unclear, though not ignorant. At any rate the Musical Means of the epimores neatly tie everything together.

THE GEOMETRIC MEAN

The GM makes the black sheep among the classical means, since it generates irrational ratios. Although the mathematicians acknowledged it, most tuners avoided it.
The GM forms the gateway to the whole subject of musical temperament. Its structural characteristics pervade the world of irrational ratios.

The formula for the GM is simple enough. In the abstract \( a/b \) equals \( b/c \). The solution gives \( b \) equal to the square root of \( ac \). In a sense the GM is the simplest of the three means. Just take the square root of the ratio. When describing such intervals in cents, just divide it in half. The GM divides the musical interval into a sonic half. As such it is necessarily a symmetrical harmony. Essentially the GM acts as the mean or mediator between the HM and AM. It sits halfway between them, as it to neutralize their difference and make a compromise in their place. We can best get a flavor of the GM by again following the Epimoric Ratio Series.

The octave 2/1 (1200c) generates the Musical Means 3/2 (701.955c) and 4/3 (498.045c), the pitches G and F. Between them sits the GM at 600c, defined as the square root of two. It's generally called a tritone, a name that implies three wholetones—C D E F#. But it isn't really an F#. Rather it is a bit higher in pitch, the 5-L version of F# belonging ratio 45/32 (590.224c). As I indicated earlier, one cannot make the exact interval in arithmetical ratios. It makes a "void point" that can only be simulated with high prime numbers, e.g. the 11-L ratio 99/70 (600.088c). On the circular graph it divides the circle in half. The interval is particularly prominent in the 12-et system, where it dominates advanced harmony. This GM can also be called 2-et because it divides the octave into two sonically equal steps.

The fifth 3/2 generates the Musical Means 5/4 (386.314c) and 6/5 (315.641c), in pitches E and Eb. Between them sits what can be called the pitch E semi-flat at 350.977c). As a ratio this GM is found to be the square root of six divided by two. This neutral third has the 11-L ratio 11/9 (347.408c) in the vicinity. Here the musical fifth is divided equally into two neutral thirds that replace the major and minor third. The major chord (C E G) and minor chord (C Eb G) are displaced by the neutral chord (C E- G).

The fourth 4/3 generates the Musical Means 7/6 (266.871c) and 8/7 (231.174c), in pitches D# and D (meaning D semi-sharp). Between them sits the ratio two divided by the square root of three, in cents 249.022c. In pitch we can call it \( \text{D#'s} \). We can also call this unusual interval a wholetone plus a quartotone.

The major third 5/4 generates the Musical Means 9/8 (203.910c) and 10/9 (182.404c), in pitch D and \( \text{D#'s} \). Between them sits the GM ratio defined as the square root of five divided by two. It makes a famous and much used interval called the mean tone, in cents 193.157c. This construction is the basis for a widely used old European tuning system that combines 5-L ratios with irrationals.

The minor third 6/5 generates the Musical Means 11/10 (165.004c) and 12/11 (150.637c). Between them sits the GM defined as the square root of six divided by the square root of five, in cents 157.821c. It becomes increasingly difficult to make an unambiguous notation for such pitches. Let's stop the epimoric sequence here.
The GM is a common feature of tempered tuning systems. Take the 12-et system. The semitone (100c) is the GM of the whole tone (200c), while the whole tone is itself the GM of the major third (400c), itself the GM of the minor sixth (800c). Again, the minor third (300c) is the GM of the tritone (600c), which is the GM of the octave (1200c). The system thus has a number of interlocking GM relations. 12-et is not alone in this regard. One can find many GM relations in other temperaments, such as 19-et or 31-et. The lesson here is simple. Tempered tuning systems accentuate the inventory of symmetrical harmonies, in opposition to arithmetical systems that do not.

**SYMmetry**

Even within a tempered tuning system the set of symmetrical harmonies remains a minority in the overall inventory of the system. For this reason symmetrical harmonies hold a certain fascination. They have traditionally been afforded a special status by the mathematicians, who love the sport of finding and describing them.

The first and simplest symmetrical harmony is the octave 1:2. By now the reader should appreciate that the octave has many special features that lead to its reputation as the 'miracle' interval of music. More on this later.

The first 3-L symmetrical harmony is also famous. It is the classical *Mousike*, the sequence 6:8:9:12. It makes the 'up' scale C F G C and the 'down' scale C G F C. The double colon marks show the point of symmetry. Here the circular graph excels, not only in confirming the presence of symmetry but also indicating the axis of symmetry, whose position is shown in the sequence by the double colon. Note on the matrix that this harmony makes a line of three pebbles (F C G).

It is the symmetrical nature of this sequence that makes it interesting and significant, not the ignorant and absurd notion that it is the only possible monochord division. Various Hellenistic writers, wanting to remake Pythagoras as the 'saint' of music, claimed that he discovered it, along with just about everything else relating to musical tuning. But in all likelihood Pythagoras acted only as a spokesman for the 3-L school of tuning—a conservative reaction against the progressivism of the Milesians.

Of course, the Mousike is not the only 3-L symmetrical sequence. Take the harmony 72:81:96:108:128:144, the monochord sequence C Bb G F D C. On the matrix it makes five pebbles in a row. Again, consider the longer sequence 432:486:512:576:648:729:768:864 which makes the monochord pitches C Bb / A G F Eb D C. Here we have seven pebbles in a row. These two sequences make prominent scales. The nine pebble symmetrical expression uses the octave 5184:10368.

Moving on to 5-L harmony, symmetrical sequences can also be found. The sequence 15:18:25:30 (C A Eb C) shows symmetrical expansion on the minor third axis of the tri-axial matrix. The sequence 20:25:32:40 (C Ab E C) does the same for the major third axis. The sequence 30:36:40:45:50:60 (C A G F Eb C) fleshes out the sequence above ruled by 15:30. But the first really prominent symmetrical 5-L harmony
is ruled by the double 60:120. It goes 60:72:75:80:90:96:100:120 (C A Ab G F E Eb C). Here all of the 5-L consonances are included in the set. Note that they form a hexagon around the generator on the tri-axial matrix, a prominent triple-strand harmony. Ptolemy used the monochord 60:120 as his standard division. Other interpolations were expressed in sexagesimal (base 60) fractions. In this way he paid homage to Babylonian (base 60) mathematics and also confirmed that 5-L harmony forms the mainstream. Base 60 arithmetic is ideally suited to monochord work since one can clear fractions for 5-L ratios.

Some prominent traditional scales also form symmetrical harmonies. One example is the sequence 90:100:108:120:135:150:162:180 (C Bb A G F Eb D C), a dorian-type diatonic scale. Another is the scale (popular from India to the Balkans) that goes 120:128:150:160:180:192:225:240 (C B Ab G F E Db C) with its ratio 128:225 (976.537c), the 5-L approximation of the septimal ratio 4.7 (968.826c).

I confess that during the '70's and '80's I also fell prey to that typical mathematician's obsession for finding symmetries. One method that I used was to plot the harmony on the circular graph and observe if it is symmetrical. Then I learned of another method from the writings of Ernst McClain, who very much influenced me during the '80's. The method is illustrated by two diagrams in this collection: the prominent 5-L monochord 24:48 and the complex 5-L division 4320:8640. One superimposes the 'up' scale over the 'down' scale on the matrix to make a combined field. Then in the place where they overlap we see a 'zone of invariance under reciprocation,' in other words, a symmetrical harmony. In the case of the 24:48 division the overlap gives only the Mousike. But in the 4320:8640 division the invariant zone forms a complex triple-strand harmony. What I found so fascinating in McClain's work, confirmed in my own investigations, is that divisions that have large overlapping zones tend to use 'cosmic' numbers found in the old mythologies (e.g. the division 360:720 or 1080:2160). The hierarchy among numbers generated by the musical context reflects the old cosmological numbers.

One can find symmetrical 7-L harmonies, but it becomes more and more difficult as we move up the prime number series. In other words, the integration of higher primes deconstructs symmetry and increases overall asymmetry. 5-L harmony has less overall symmetry than 3-L harmony, 7-L harmony still less, and so on. One chooses n-Limit harmony if one desires minimal symmetry. The Harmonic Series is maximally asymmetrical.

All of the symmetrical harmonies given above have something in common. They all use the generator tone C as the point of symmetry. But this need not be the case. Let's use the Mousike as a simple example. On the matrix it forms the pebble pattern F C G. If we now move the pattern one spot to the right we get C G D, another mode of the same harmony. Now the point of symmetry sits on G. It has the monochord sequence 9:12:16:18 (C G D C). Three modes are possible with this harmony. Obviously, a symmetrical structure with seven elements has seven possible modes, all of which put the
point of symmetry in a different place in the scale. In order to cover all of the possibilities I use the general term point symmetry.

However, point symmetry is not the only form of symmetry. Already in the 1970's I observed what I call axis symmetry on the circular graph. In these structures the axis of symmetry runs through the harmonic axis—the fifth C G or fourth G C. During the '80's I largely ignored or downplayed this form of symmetry. I gave it secondary status, being under the influence of McClain. However, it is just as valid as point symmetry.

Here is a simple example of a 3-L harmony with axis symmetry. The sequence 24:27:32:36 can have the 'up' scale C D F G and the 'down' scale G F D C. Here the sequence is bound by the fifth (C G) rather than the octave. As an example of a 3-L symmetrical sequence bound by a fourth, take 216:243:256:288. It makes the 'up' scale G / A Bb C and the 'down' scale C Bb / A G. One can also find such forms of symmetry in 5-L harmony. For example, the sequence 20:24:25:30 gives the 'up' scale C Eb E G and the 'down' scale G E Eb C. Using the fourth, the sequence 45:50:54:60 gives the 'up' scale G A Bb C and 'down' scale C Bb A G. Many more such examples could be given.

As we have seen, a major third can also generate a symmetrical pattern, where 72:80:81:90 gives C Bb A Bb A. Indeed, any union of the HM and AM makes a symmetrical sequence. However, in practice only point symmetry and axis symmetry has much sway. As I have grown older I have become more wary of assigning special status to symmetrical harmonies. After all, the most useful of harmonies, such as the major triad, are not symmetrical. Perhaps we should curb the mathematician's appetite for symmetry.

THE CLASSIFICATION OF HARMONY

Two fault lines serve to delineate a classification of the myriad possible alternatives. A given system uses either arithmetical ratios or irrational ratios, and a given system is either open-ended or forming a closed loop (generally a circle of fifths). These parameters lead to a simple scheme of four principal types. However, the reality is not so straightforward. Some tuning systems mix rational and irrational ratios. Even equal-tempered systems like 12-et that purport to use all irrational ratios still employ one rational ratio, the octave. Meantone systems include some more rationals, but use mainly irrationals. One can manipulate the balance between rationals and irrationals in various ways. Thus a more sensitive classification of harmony must not restrict itself to a simplistic either-or position.

I classify harmonies that use only arithmetical ratios as forms of just intonation. Unfortunately, for many theorists the term refers only to 5-L harmony. I use it to mean everything from 3-L harmony to the n-Limit, that is, the whole world of arithmetical ratios and their number sequences. Meantone systems then form a special sub-group of the temperaments. Another sub-group consists of the equal-temperaments, also called
‘multiple divisions’ that maximize the irrationals. Such systems are necessarily regular, meaning that all the fifths in the loop are the same size. However, this need not be the case, defining another sub-group of temperaments called irregular. Here one sees different sizes of fifths in the loop. A special subset of this massive group has special properties that make it a well-temperament or ‘circular temperament.’ In short, the notion of temperament requires various refinements.

The classification of tuning systems as either open or closed is also fraught. Sometimes just the smallest adjustment transforms an open system into a closed circle of fifths. In that case the two systems sound like each other and have a similar morphology, even though they sit in opposing families. Here the classification seems largely meaningless—two avenues for describing the same sound-world. Yet we keep this classification because not all open systems are so cleanly altered into closed analogs. Again, the situation is complicated enough to make it worthwhile that we preserve the notion of open and closed systems.

Forms of just intonation exemplify the very heights of openness. Such systems use integral relations derived from a single source, the 1/1. They are open-ended because they proceed from simplicity to some desired level of complexity in the ratio relations, a level that can always be exceeded. Some ‘arbitrary’ index of expansion (e.g. a double) must be imposed, since numerosity must be curtailed. Wherever the limit is placed, the resulting harmony will define some field of relations (a matrix) that can always be functionally or vectorially related to the unity, the ‘tonic’ or the source tone.

Systems of just intonation can best be classified by the Prime Number Series. 3-L harmony, that open-ended line of fifths, is often called Pythagorean, as if Pythagoras discovered it. 5-L harmony, the open tri-axial matrix, would sometimes be called tertial or just plain ‘just intonation.’ 7-L harmony is often called septimal. And so we move up the series of prime numbers, adding new forms of dissonance until we saturate the field. All forms of just intonation are open systems.

Meantone systems keep some just ratios but temper segments of the line of fifths. As such they are also open-ended like 3-L harmony. The well known, most used, and arguably the single best meantone system is certainly one-quarter (syntonic) comma meantone. Here is its rationale. When we tune four fifths (C G D /A /E) we end up with the dissonant third C /E (ratio 81/64). But we want the just third C E (ratio 5/4). The solution is to squeeze each fifth by a quarter comma (5.3775). Then the D will sit at one-half comma flat of a 9/8, which is the same as one-half comma sharp of D (10/9). This is the official meantone, the GM of the 5/4. The A will be three-quarters of a comma flat of /A (27/16), which is the same as one-quarter comma sharp of A (5/3). As a result the thirds and sixths are quite good, with the burden of the temperament conveniently falling on dissonances like the wholetone. The adjustments sit in fortunate positions. The classical meantone scale uses a line of fifths from Eb to G#. The line is open, so that a fifth from G# would need D#. The system preserves the enharmonic distance between D# and Eb. Indeed, our modern notation of sharps and flats was invented for it. Using the fifth G# Eb instead of G# D# results in a dissonant, diesis-raised fifth often called a ‘wolf’
fifth. Thus the system is only artificially restricted to twelve notes—it tends to expand. The long line of meantone fifths almost closes at thirty-one members. With very minor tweaking it can be transformed into the closed system 31-et. In other words, 31-et offers the same sound world but with the added feature of unlimited transposition.

This excellent system is not the only form of meantone tuning. Here are a few other examples, showing a similar type of rationale. When we tune three pure fifths (C G D /A) we get the dissonant major sixth /A (27/16), but we want the consonant A (5/3). The solution—temper each fifth flat by one-third of a comma (7.169c), almost a diisma. This one is not as successful, since the tempering of the fifth is arguably excessive. However, it still has some special unique features that make it worthwhile. It can be transformed into a closed circle of fifths by using 19-et. As another example, consider one-fifth comma meantone. When we tune five pure fifths (C G D /A /E /B) we get the dissonant /B (243/128), but we want instead the less dissonant B (15/8). The solution is to adjust each fifth flat by one-fifth of a comma (4.312c). It has a closed analog in 43-et. Another useful example is one-sixth comma meantone. Tuning six pure fifths (C G D /A /E /B /F#) gives /F# (729/512), but we want F# (45/32). Hence we squeeze the fifth by one-sixth comma (3.584c). This system has a closed analog in 55-et.

A lot more open meantone systems can be specified by complexifying the rubric. Every one of them has a ‘cousin’ analog in a closed cycle equal-temperament. For example, two-sevenths comma meantone is like 50-et, three-tenths comma meantone is like 69-et, and two ninths comma meantone is like 74-et. Every meantone system has a different individual character, but each one displays a form of ‘meantone’ that lies between D and D.

Does it not seem odd that an open system like a meantone temperament can be finely mimicked by a closed cycle equal-temperament? These multiple divisions have an entirely different rationale. But the overlap is not one-to-one. Not every multiple division is related to a meantone system. Indeed, only a minority has this property. In fact, equal-temperaments obey another rationale that has nothing to do with the line of fifths. Divide the octave into sonically equal sized steps. In such systems every ratio except the octave is irrational, and expresses some root of two. For example, in 7-et the step interval is the seventh root of two. In 12-et it is the twelfth root of two. One can proceed up the number series and investigate each multiple division in turn. Each must be individually assessed. The majority of them are either really bad or mediocre. Sometimes they are just passable. But every so often one encounters a system that is really good, such as 12-et or 31-et or 53-et. Each division makes its own harmonic landscape.

Nevertheless, these divisions can still be theorized as alterations of the 3-L line of fifths. Sometimes the tempered fifths form a closed loop—a single circle of fifths. But in other multiple divisions we see multiple circles of fifths. For example, 24-et has two circles of fifths. They are not confined to a single-circle base. Equal-temperaments display an incredible variety of structural characteristics, making them very interesting for tuning theorists.
The set of closed-cycle equal-temperaments is open-ended. One can always make a higher resolution division. However, we inevitably leave the realm of practicality and enter an ‘atomic’ sphere that increasingly approaches the subliminal. The law of diminishing returns sets in. Nevertheless, one can still assess the value of a high resolution system. Take this example: 441-et is vastly superior to 440-et or 442-et. Even in such high resolutions certain systems stand out as special.

While investigating multiple divisions I discovered that every equal-temperament can be mapped as a unique alteration of the tri-axial matrix, thereby creating a ‘tile’ with boundary zones and an axis of deviation (one side tempered flat, the other side tempered sharp). This map clarifies the structural characteristics of the temperament.

CIRCLES OF FIFTHS

The line of 3/2 fifths goes on forever, but by mistuning the members in subtle ways we can change it into a closed loop of fifths, an irrational system. Some mistunings work only poorly, but others work very well indeed. An excellent exemplar sits in 53-et. If we tune a cycle of fifty-three pure fifths we come upon a microtone of about 3.6c above the thirty-first 2/1. This small error must then be distributed between the fifty-three members, giving a fifth that is tempered flat by only about 0.07c. This value is well below the hearing threshold. Consequently we can judge 53-et an accurate simulation of the 3-L line. All the inherent varieties of 3-L harmonies preside in it. Of course, 53-et is yet more remarkable, since it’s also the bed of an excellent simulation or embodiment of 5-L harmony, a passable 7-L, but a mediocre 11-L. The 53-et system has more features, but it can certainly be theorized as a single closed circle of fifths.

Of course, 53-et is not the only, let alone the best such cycle. An even more excellent exemplar sits in 306-et, where the error amounts to about 1.8c below the 179th octave. Here the fifth must be tempered sharp by about 0.01c. This one is extremely close to the 3-L line. It also holds the distinction of being in the family of 612-et, the schisma resolution. I judge 612-et a special resolution because it offers a fine-grained view on the subtleties of just intonation as well as temperament. A clisma is 4 schismas in size, a ditonic comma 12, a syntonic comma 11, and so on. 306-et forms a double-schisma or semi-clisma resolution. The family of 612-et has a good number of interesting sub-sets. One of them is 12-et.

When we tune twelve 3/2 intervals the error is a ditonic comma (23.460c). This interval happens to be 12 schismas in size, so each fifth must be tempered flat by a schisma (1.961c), called a schismatic fifth. In this way the line of fifths becomes a closed loop, the 12-et circle of fifths. Yet 12-et is also a member of the wider ‘galactic’ family of 612-et. The 12-et semitone equals exactly 51 schismas.

The best well-temperaments also belong to this family. They play with the schisma resolution by tuning some fifths pure, some schismatic as in 12-et, and some two schismas flat—in the range of meantone. This method results in an irregular temperament that beautifully distributes the error in such a way that we have both meantone-like and
pythagorean-like chords. The common keys, like C or G sound closer to meantone in character, with tempered fifths and better thirds. The distant keys like Gb or Db sound closer to pythagorean with perfect fifths and sharper thirds. The transition between them is smooth. The twelve major chords and twelve minor chords all have different "colors."

Well-temperaments can come in many varieties, but they are all irregular variations of the 12-et regular circle of fifths. Instead of distributing the error evenly as in 12-et, it sits a little more here and a little less there. The tolerances are quite fine, so that one can modulate into every key and yet every key has its own character. In 12-et every key is the same. This tuning philosophy arose during the baroque era with the discovery of logarithms, liberating the control of temperament. Many baroque tuners viewed the schisma as a unit of temperament—an appropriate term. But during the 19th century when technicians decided on the cents system (1200-et) they ignored the old sensibilities and went for something conveniently related to 12-et. The cents system is basically a semi-schismatic resolution, but not a very good one. One could do better with 1224-et, the double of 612-et. No matter, measurements can easily be translated between systems.

The theoretical model of a circle of fifths proves to be quite fruitful. We have already seen it at work in the meantone group, where quarter-comma meantone has an analog in 31-et. This enharmonic resolution can also be modelled as a single circle of fifths, each fifth tempered a little. In theory any regular multiple division, such as 31-et, can be used as the basis for many forms of well-temperament with different sizes of fifths. However, historically it has never been done. All proposed well-temperaments refer back to 12-et.

As we have seen above in 306-et, the fifths can be tempered sharp as well as flat. This method is perhaps psychologically more fraught, but still one can find good resolutions. For example, in 41-et we have the analog of a cycle where forty-one pure fifths makes an error of about 19.8c below the twenty-fourth 2/1. Each fifth must be sharpened by 0.48c, quite good. But this system is no longer a meantone related harmony, it has both a D and a DB. It is thus more akin to 53-et, which is definitely not a meantone harmony. The lesson in all of this is that the set of multiple divisions groups itself into a number of families or types.

Moreover, many equal-temperaments don't even make a single circle of fifths. They may make multiple circles. For example, 72-et embodies six self-enclosed circles of fifths that stack on the matrix. It presents a very different morphology than meantone, more a 'squeezed comma' resolution that is like a variant of 53-et. But it also differs from 53-et in various ways, for instance showing a lot more symmetrical structures and having 12-et as a subset in the family.

I used to negatively judge systems with multiple circles of fifths, calling them 'fragmented' rather than 'integrated.' To be sure, most of the best multiple divisions show one circle, but I have become wary of pre-conceptions. One can also find quite good systems, like 72-et or 34-et (a subset of 612-et) that are 'fragmented.' Moreover, it's not hard to find mediocre systems, like 29-et, that have one circle of fifths, or even
poor ones like 33-et. The single circle of fifths theoretical model is useful, but it has its limitations. The set of multiple divisions is not bound by it.

THE SCHISMA

The schisma (s for short) is a remarkable interval, feeling quite at home in both contexts—temperament and just intonation. Moreover, it sits on the edge of the subliminal or just over it, a veritable musical atom or smallest building brick. When I drew up the diagrams I had already realized its importance as the connector between 3-L and 5-L harmony. But I did not yet appreciate its wonderful utility. It simplifies the numbers and makes micro-structural issues clear. I decided, for this essay, to use the cents measurement consistent with the old diagrams. Most of the ratios in this paper are expressed in cents (1200-et). However, in this section I want to illustrate how illuminating it can be to reference 612-et instead of double 600-et. The numbers lose some of their arbitrary appearance and become more functional as well as simpler. Thus it aids understanding.

The schisma resolution makes the description of tolerances clear and simple. How far can we temper the fifth before it becomes too rough? Most people agree that it sits around 3s or maybe between 3s and 4s. The clisma (4s) is definitely over the top. 3s is equal to one-quarter of a ditonic comma (12s). The meantone version of one-quarter of a syntonic comma (11s) yields 2.75s, safely inside the comfort zone. One-fifth syntonic comma gives 2.2s, one-fifth ditonic comma 2.4s. One-sixth syntonic comma equals about 1.83s, one-sixth ditonic comma 2s. Looking on the wide side, one-third syntonic comma gives 3.67s, one-third ditonic comma 4s. The best meantone temperaments deal with less than 3s.

So much for tolerances in meantone temperaments. We now leave the realm of temperaments and turn to the measurement of commas in the just intonation fabric. Here the schisma resolution shines. The 5-L diachisma-comma (ratio 2048/2025) yields 10s, the 5-L syntonic 11s, the 3-L ditonic 12s. Moreover, the 7-L comma (ratio 64/63) gives 14s. The 5-L so-called ‘acute’ comma (ratio 3125/3072) measures 15s. Notice that something interesting is happening between these schisma numbers. 11s=10+1. Again, 12s=11+1. Moreover 14s=10+4, while 15s=11+4. This odd additive rubric continues on. The 5-L diesis or enharmonic (ratio 128/125) sits at 21s, that is 11+10. We can safely call it two commas in size. Archytas used the 7-L diesis of ratio 36/35, which sits at 25s, that is 21+4 or 11+10+4. It transpires that all of the significant larger intervals can be constructed out of a few primary bricks: 1, 4, 10 and 11 schismas in size. Let’s look at some examples of small intervals up to the wholetone.

Take the chromatic semitone of ratio 25/24. It equals 36s. We can describe it as three ditonic commas 12+12+12, but another way can be 11+11+10+4. Whichever path we choose it amounts to three commas. Archytas used as his diesis the 7-L ratio 28/27, which is 32s in size. That is 36-4, but also 11+11+10. The large 5-L diatonic semitone of ratio 16/15 is 57s in size. That’s a chromatic semitone plus a diesis, 36+21. Here we have a semitone of five commas that can also be derived as 11+11+11+10+10+4. The 3-L
apotome of ratio 2187/2048 is 58s in size, the schismatic diatonic semitone. It can also be constructed as 11+11+11+11+10+4.

The medial semitone, which is four commas in size, also follows the rubric. For the 5-L medial semitone of ratio 135/128 equals 47s, a chromatic semitone plus a syntonic comma or 36+11. It also equals 11+11+11+10+4. Meanwhile the 3-L medial semitone or limma, of ratio 256/243 yields 46s, that is 47-1, making it a schismatic medial semitone. We can also generate it as 11+11+10+10+4. By the way, the 12-et semitone sits at 51s, that’s 47+4. The 7-L medial semitone of ratio 21/20 equals 43s, which is 47-4. It is also 11+11+11+10. Relations between 5-L and 7-L variants always involve a clisma shift, for example between the diatonic semitones 16/15 (57s) and 15/14 (61s) or 57+4. The 5-L stretched (six comma) semitone of ratio 27/25 sits at 68s or 57+11. That is also 11+11+11+10+10+4. Archytas used the 7-L stretched semitone of ratio 243/224. It sits at 72s, or 68+4. At this point we have covered all of the most significant semitones.

The 11-L three-quarter tone ratio 12/11, used by Ptolemy, equals 77s. It can be seen as seven commas, in fact seven times 11s. The 11-L ratio 11/10 sits at 84s. That is 11+11+11+10+10+10+10, a squeezed version of eight commas. Moving on to wholetones, the 5-L wholetone of ratio 10/9 sits at 93s. That can be decomposed into the pattern 11+11+11+11+11+11+11+4+1 or eight commas. The 3-L wholetone 9/8 gives 104s, or 11+11+11+11+11+11+11+4+1 or nine commas.

I could go on but I have made my point. From a schisma resolution perspective the field of 3-L, 5-L and 7-L harmony, and even 11-L harmony, consists of a bed of clisma and shisma shifts. We could get another perspective on it by using a clisma resolution like 152-et or the 612-et subset 153-et. In my own practice I have found that the best insight on the deep micro-structures of the just field lies in the schisma perspective. However, for practical reasons, it simplifies into the comma resolution 53-et is close enough for a workable overview of practical just intonation. Of course, if we also want exotically high prime numbers and a field approaching saturation, then all bets are off. Then 612-et no longer suffices and we end up needing lots of decimal points, the way we do in the cents system. No logarithmic perspective, even 612-et, will do justice to the fantastic complexity of n-limit arithmetical ratios.

THE CITY WALL

Most of my collection of diagrams concerns just intonation, reflecting my bias during the 1970’s and ‘80’s. However, I made one diagram that combines a number of equal-tempered divisions. Such systems collapse the open 5-L matrix into a ‘tile’ or closed system with boundaries. In the case of 53-et, the schisma between the syntonic comma (21.506c or 11s) and ditonic comma (23.460c or 12s) is tempered out of existence by using a ‘mean’ comma (22.641c or 11.547s). It stands for all the variety of commas that we see in the fabric of just intonation. The scale is then conveniently collapsed into a scale of commas that can also be expressed as a single circle of fifths. In this section I want to explain why the boundaries of the 53-et tile sit where they do.
My commentary concerns the diagram called *Matrix*, an example of a wide expansion of the 5-L field. The boundaries of 53-et make a pattern I sometimes called ‘the flag,’ because it looks like a flag waving in the wind. Another appropriate and more fertile metaphor is ‘the city.’ I took this description from early Greek philosophy (Heraclitus and others). A harmony is like a city because it consists of a finite number of citizens who have relations with each other. The city metaphor proves to be evocative. In the 53-et context the middle area is the ‘downtown,’ surrounded by two ‘suburbs’ defined as comma-raised and comma-lowered. These three regions are separated by the comma boundary. The consonance hexagon makes the ‘centrum’ with ‘C’ as the heart, the temple to the goddess. The two suburbs have their own hexagons centered around C to the west and 7C to the east. This assemblage is surrounded by the city wall for 53-et, shown by the thick line.

The diagram lays out what happens when we tune in pure fifths and thirds. Tuning far to the west or east results in pitches that are schisma-altered versions of the pitches inside the city wall. I have marked schisma alteration by a line above or below the pitch name. I call this east-west boundary the schisma boundary. In some places I have also called it the eastern and western gateway. It sits in this position because it makes an even expansion from the central 1/1, and because this position employs the smallest ratio numbers to have this relation. Hence it acts in an analogous way to the pitches D and 7D, the smallest numbered ratios to give the comma relation and hence the comma boundary.

Here is a specific example of the workings of the schisma boundary. Tuning west from C along the line of fifths we come to Db just beyond the wall. It has a schisma-lowered mark to distinguish it from 7C#. It is found just inside the wall across the city. The pitch Db is the 3-L four-comma (limma) semitone of ratio 256/243 (90.225c or 46s) while the pitch 7C# gives the 5-L version of ratio 135/128 (92.179c or 47s). The 53-et system eliminates this schisma, producing a compromise four-comma medial semitone that can be called both Db and 7C#. Remember that C# has three commas, Db five commas, so the pitch between shows comma alteration as 7C# and Db. My notation of pitches is based on function—how the pitch relates to the reference ‘C.’ Unfortunately, such pitches that straddle the city boundary must have more than one name. In other words, forming a temperament with its closed wall leads to functional ambiguity. In pure 5-L harmony this ambiguity does not exist and I could call the disparate pitches Db and 7C#, but in 53-et these two functions become ‘homogenized.’ A similar situation ensues with all the pitches that straddle the wall. For example, in 53-et the 3-L pitch 7B just outside the city is the same as 7Cb just inside the city.

The same principle applies to the north and south wall. If we tune pure along the minor third axis C A 7A Db# we cross the wall. But this pitch is almost the same as the pitch Ebb, found in the south just inside the wall. They differ by a 5-L clisma—shown as a dot above or below the pitch. Again, the 53-et system tempers away this difference so that one pitch (eleven commas in size) can be called both Db# and Ebb. This tempering creates the north-south wall, which I sometimes called the north-south gateway, more properly the clisma boundary.

34
This eleven-comma interval has a further significance. It is the GM of the fourth C F (twenty-two commas in size). The GM of the fourth G C sits at \( \text{A} \)# (equals /Bb). These two pitches \( \text{D} \)# and \( \text{A} \)# form the Harmonic Antipodes of the 53-et system. This term means that they sit at the opposite end of the circle of fifths from the generator tone C. Every system with a single circle of fifths has a harmonic antipode. For 12-et it is the tritone, the GM of the octave. It defines the most esoteric component of the system. Unfortunately, in the diagrams of that time, I used the description ‘harmonic antipodes’ for another meaning entirely—a blatant shift of terminology between then and now. There I referred to what I would now call the ‘four corners region’ of the 53-et city wall, which also has certain special characteristics (an approach to 11-L values). This is the only change of terminology that I have found in the old diagrams.

The Matrix diagram also shows the region represented by 31-et and standard meantone tuning. It is the central pattern oriented around the major third axis. Observe the line of major thirds Bb D F# and the line \( \text{Gb} \) D H#. In meantone and 31-et they become the same pitches, being altered by half a comma. I call this double column the meantone corridor, an expression of the comma boundary. The temperament thus tempers the comma out of existence. Between this meantone corridor can be found the materials of 31-et. The line of thirds Eb G B is tempered one-quarter comma flat, the line Db F A one-quarter comma sharp. The central line Ab C E is, of course, tuned pure. Thus we see the orientation of this tuning system.

I have also indicated the boundary wall for 19-et and one-third comma meantone. The pattern orients along the minor third axis. In this system the line of minor thirds Eb C A is tuned pure, the line Bb G E one-third comma flat, and the line Ab F \( \text{D} \) one-third comma sharp. Thus, for example, the 19-et system has no Db. In order to get a large diatonic semitone one must use a tempered version of /Db, the stretched diatonic semitone. Thus the 19-et system whole tone divides into a small (chromatic) semitone and an over-sized diatonic semitone. This morphology is characteristic of 19-et and its cousin one-third comma meantone.

Also on the diagram one can specify the part of the matrix approximated by 12-et. It is the central line of fifths from \( \text{F} \)# across to \( \text{G} \)#. A symmetrical expansion of the line of fifths results in two tritone separated by a ditonic comma. Hence the GM of the octave, the tritone, is the harmonic antipode of the system. The 12-et system is true to its 3-L orientation. The fifths and fourths sound practically pure (schismatic) but the thirds and sixths take the burden of the temperament. The 12-et major third is 7s sharp, almost a double clisma. The major sixth sits 8s sharp, a full double clisma. The complaints with 12-et have always come from the mediocrity of the thirds and sixths. On the other side the praise of 12-et has always been based on the superlative value of its fifths—only compromised by one schisma, not much. Since fifths have a higher consonance status than thirds, better to give preference to the fifths and let the thirds suffer. So goes the common argument in favor of 12-et.
Finally, on this diagram, I have also drawn in the boundaries for the clisma resolution, 152-et. It is that large pattern that includes much of the diagram. Unfortunately, the city wall of 612-et lies well beyond the scope of this diagram. You can see it on my website: www.siementerpstra.com. The selection of resolutions that I had chosen for this *Matrix* diagram also reflects my present values. I tend to model harmony in schismas, clismas, commas, and dieses. However, this particular group is not meant to be exhaustive. Every multiple division has its own field boundaries, its own functional ambiguities, and its own character or ‘ethos.’ In contemplating a clisma resolution I could have chosen 171-et instead of 152-et, also a good one. But I have chosen not to put the boundaries for 171-et on this diagram. Also, in the diagram I decided to ‘hide’ the axis of deviation for the resolution, in the interest of reducing clutter and producing a more pleasing image.

**ARISTOXENUS AND COUNTER-ARGUMENTS**

I cannot leave the subject of temperament without a consideration of that extraordinary pupil of Aristotle. Aristoxenus abandoned arithmetical ratios altogether in favor of a geometrical approach. Instead of ratio number sequences, he saw the octave divided into steps or ‘parts’ (p for short) that are presumably sonically equal in size. In this way he could use the sort of additive language that we associate with temperament. For example, he says that the wholetone (12p) can be equally divided into half-tones (6p) and quarter-tones (3p).

His language is a consistent and accurate description of 72-et. However, his heresy has provoked a strong response, not only in ancient times but even in modern disguises. Most writers contend that he either could not have done it, or else that he should not have done it, or both. A series of arguments have repeatedly been put forth, attempting to prove that Aristoxenus could not have implemented 72-et.

One argument claims that he was confused or incompetent. He didn’t know what he was doing. However, this argument has no weight because he represents the most intelligent of all the ancient writers on the subject. Not only did he write about musical tuning, but also rhythm, instrumentation, history and other topics. We have more real information from his book than any other single ancient source. Obviously, he knew what he was doing.

Another argument contends that he could not have used equal-temperament because neither he nor his followers ever mentioned the term ‘equal-temperament’ or another term equivalent in meaning. True—he did not call it anything. But so what? It nevertheless proves to be an accurate description of 72-et in every respect.

Another argument I would call sociological. Ancient Greek music culture (like the middle-east and India) was entirely monophonic and heterophonic. They had absolutely no need for a tempered scale. Such approaches only become appealing in a context of modulation, chord progression, transposition, and so on. Since the Greeks had no context for temperament, Aristoxenus is unlikely to have proposed one. This is true, but it ignores
his scientific stance. Aristoxenus wants to comprehensively map a continuous linear space (the monochord) with variable quantities. He wants a practical simulation of the continuum. As such he wants something like atomism—very small, sonically-equal steps that can aggregate into larger intervals to be measured additively. Hence his aim was remarkably similar to our modern aim in using the cents measure. It finds kin with the schisma measure and the comma measure. His system of parts acts as a metrical agent and he saw advantage in this perspective. Alas, Aristoxenus has never been given proper credit for this valuable insight.

Another widespread argument contends that Aristoxenus abandoned ratios because he was totally opposed to any exact numeration. Instead we should tune intuitively ‘by ear’ alone. Empirical experience stands opposed to theoretical number calculations. But this argument also falls down and proves to be another attempt to make him look incompetent. Aristoxenus only abandoned arithmetical ratios, but not geometrical ratios, which are equally exact. The dichotomy between exact arithmetical ratios and supposedly ‘vague’ intuitional irrationals is an absurd fantasy. Aristoxenus simply proposed an exact system of irrationals. Moreover, the supposed reliance on the ear alone is generally derived from a single passage that is then taken out of context. Aristoxenus said that his tempered fourth (one schisma sharp) is acceptable because it is sufficiently close to the just fourth. Is this not true? His detractors take this to mean that all approximations are good enough and we can abandon ratios and accuracy altogether. Again, it betrays another effort to make Aristoxenus look like a fool.

Another argument says that he could not have proposed an equal-temperament because it is simply impossible to set such a tuning on a monochord. But this statement is only true if we stick to the traditional arithmetical method. Irrational numbers generate never-ending decimal points, so they cannot be accurately expressed in whole number ratio sequences—except in approximation. In order to make these approximations decently good the numbers must become quite large (theoretically infinite). However, this whole issue is beside the point, because Aristoxenus did not use arithmetical sequences. Instead he used a geometrical construction to find his fret positions.

We can confirm that systems like 31-et and 53-et would have been beyond ancient capabilities, since they had no logarithms to deal with multiple roots of two. An old problem then remains—how did Philolaus know that the whole tone equals nine commas in size? I propose that he used what I call the rough method of monochord work. This explanation is conjectural but based on common sense and many years of experience with monochords. If Philolaus had the calculated fret positions for the first fret (comma), the fourth fret (limma) and the fifth fret (spatome), he could have ‘eye-balled’ frets two and three, sidestepping any calculation whatsoever. Remember that the monochord always needs a little tweaking anyway, so this manipulation is not so unconventional. Tuning theorists tend to ignore such procedures, since they obsess on the numbers, but on a practical level one should not ignore these possibilities. Moreover, we have circumstantial evidence that Philolaus took this route, for he did not give the numbers for an enharmonic genus. Also, his chromatic genus is not very true to form, since it uses
large semitones. Indeed, the numbers for a 3-L enharmonic division, or even a proper chromatic division, prove to be extremely difficult. We face the mega-number problem.

The first extant writer to give numbers for all three genera was Archytas, and he used the 7-L. It was as if he thumbed his nose at the long established 5-L norms and made a virtuoso calculation. Aristoxenus must have been thoroughly impressed by Archytas, since he wrote a biography of him, now lost. But to get back to the argument, it simply is not true that no equal-tempered scale can be set on a monochord. Sure, 53-et is impossible, but Aristoxenus chose the one tempered scale that is easy to achieve using classical Greek geometrical methods.

To accomplish this harmony we need only three tools in a toolkit. The first tool is the square root of two, an extraction long known, even before the Greek era. The second tool hails as the cube root of two, a popular puzzle for ancient mathematicians. Various writers proposed solutions. Archytas himself was renowned for a construction using three-dimensional geometry. The third tool consists of the theorem of Thales that enables the construction of a GM on a given line segment. Using these tools we can proceed as follows: Take the square root of two positions. Now we have placed 3-et. Take the square root to get 6-et, another to get 12-et, another to make 24-et, and finally a cube root for the desired 72-et. Other pathways are also possible, so that one can also have 36-et. Moreover, this method is open-ended. One could further divide 72-et to get 144-et and so on. Aristoxenus (and his potential predecessors, notably Democritus) had good fortune in that the one tempered system open to exploration also happens to be a musically good system, namely 72-et and its mates.

72-et is a highly symmetrical system with ample good resources. We can observe them by looking at its small intervals, proceeding to the whole tone. First of all, it is closely affiliated with 612-et, in fact a direct subset of double 612-et. The smallest step, 1p, equals 16.667c or 8.5s, a squeezed comma or more accurately a double clisma. However, this micro-interval is ‘atomic’ and not directly used. The smallest usable interval is 2p, called a one-sixth tone (33.333c or 17s). It sits close to the minimal enharmonic (18s). The following 3p makes the quarter-tone (50c or 25.5s). This one just grazes Archytas’ value (25s). Then 4p defines a third-tone at 66.667c or 34s. It lies well within the chromatic semitone range, midway between the 5-L (36s) and 7-L (32s) forms. The interval of 5p makes a small medial semitone (83.333c or 42.5s), very close to the 7-L medial (43s). The famous 6p, usually just called a half-tone or semitone, is more specifically the large medial semitone (100c or 51s) used in 12-et. Then 7p makes the large diatonic semitone (116.667c or 59.5s), about half way in size between the 5-L (57s) and 7-L (61s) versions. 8p gives the two-thirds tone (133.333c or 68s), identical to the 5-L stretched semitone (68s). The interval of 9p defines the three-quarter tone (150c or 76.5s), close to the 11-L version (77s). Then 10p yields 166.667c or 85s, called a five-sixths tone and close to the 11-L interval (84s). 11p equals 183.333c or 93.5s, close to the 5-L whole tone (93s). Finally, 12p makes the large whole tone (200c or 102s), just a double schisma (or half clisma) flat of the 3-L norm (104s).
Thus the system has much subtlety and includes many good approximations of various important 3-L, 5-L, 7-L, and 11-L ratios. No doubt Aristoxenus meant to use it as a way of adequately simulating and comparing the tunings of his day, just the way we use our cents system. For he converted the scales of Philolaus' diatonic and Archytas' enharmonic into the close equivalents of his own system. Moreover, he gave two tunings quite close to each other and on the border between the enharmonic and chromatic genus. In this manner he showed that one can make subtle variations on the genera—what he called 'shades' of the genera. Finally, he recommends in one place that we divide the step itself in half (implying 144-et). In other words, it is permissible to deepen the resolution if the 72-et resolution is not close enough for one's purposes. Thus he clearly conceived the notion of a variable resolution system that can accord to the aesthetic context. Indeed, this valuable insight must be his greatest achievement, but he has never been given credit for it.

72-et proves to be well suited to variable resolution. After all, it has 12-et in its core. An even better way to look at it is quarter-tone with inflections. 24-et forms a convenient enharmonic scaffold that can be tweaked. I use this approach myself on my oud (Arabic lute). The instrument is like a fretless guitar, essentially a monochord except that my finger replaces the moveable bridge. I then calibrate 24-et on the side of the neck as the reference, then I place my finger optionally a little before or a little behind the quarter-tone position. It is both simple and practical, yet subtle and powerful. 24-et is still the theoretical basis for the tuning cultures of Iraq and Kurdistan.

Aristoxenus achieved much, and he left a school that persisted for hundreds of years. Unfortunately, his followers did not equal his stature, and it appears that often they did not even understand his writings. Consequently, they did not further the great potential of his approach. Instead they just copied the scales that he happened to leave, as if no other scales were possible. Thus his followers lost the whole scientific spirit of his work. It appears that many followers did not even realize that the ratios are geometrical. Over a long period of time confusion and corruption ensued. This situation only made it easier for rival schools to attack his ideas. These attacks began right away.

Euclid, who lived not long after Aristoxenus, offers a good example of a writer in the attack mode. In one theorem he proves that you can't divide a 9/8 whole tone into equal halves, the way Aristoxenus contends. He goes on to divide the nine-comma whole tone (104s) into the four-comma limma (46s) and the five-comma apotome (58s). However, he never mentions that one can also achieve these comma divisions with arguably more consonance, at least smaller ratio numbers, by using the 5-L version that goes 47s+57s. Thus he mainly displays his 3-L conservatism. Yet the whole exercise is beside the point, because one can divide the 9/8 in half, by taking the GM—104s yields the irrational of 52s. Euclid gave the geometrical procedure himself in the Elements. Thus his writing is more an anti-temperament polemic than anything else. In another example, Euclid proves that six whole tones exceed an octave by a ditonic comma, thereby throwing ridicule on Aristoxenus' observation that six whole tones make an octave. Of course, Euclid uses 9/8 whole tones (104s), but Aristoxenus intended tempered
whole-tones (102s). This is typical of the way that Aristoxenus was treated by other theorists right through the ancient period (and beyond).

Ptolemy rehashed Euclid’s arguments but he also attacked Aristoxenus in a more positive and intelligent manner. By allowing n-limit arithmetical ratios he generated complex subtleties with even more possibilities for ‘shades’ than the neat system of 72-et. He exceeded him in the complexity game. Only al-Farabi (and his student Avicenna) advanced the inventory of Ptolemy. Avicenna had a preference for 13-L harmonies. Meanwhile al-Farabi, unique among the warring theorists, gave some tunings both in ratios and in parts. He was capable of inter-translation. He used certain patterns in Aristoxenus’ system to organize the varieties of arithmetical divisions. In my opinion the 10th century al-Farabi represents the very peak of the ancient achievement. Not much more of interest happens in the field of musical tuning until the 15th century in renaissance Europe, when meantone came into fashion. Meantone proved to be Pandora’s box—out popped resurgent pluralism.

TETRACHORDS

Traditional tuning cultures used two intervals as the basic modules of scale construction. These are the octave (1:2) and the musical fourth (3:4), called a tetrachord. So far in this paper we have focussed on octave sequences, but in this section tetrachords take center stage. Tetrachords make a very practical scalar fragment or unit. They can be combined in various ways to form an octave or double octave. They also come in numerous types that may be classified as diatonic, chromatic, or enharmonic. These modules can be mixed together in a good variety of ways. The great utility of this approach has assured that tetrachords have a long history over a wide geography of music cultures. It formed the basis not only for Greek theory but also classical Indian and Persian theory. Thus the tetrachord has long been the essential scalar module.

Two tetrachords can be combined into a frame with an 8:9 connector to make an octave. It may be done in three ways, depending on where we place the connector: top, middle, or bottom. Placing the 8:9 on top defines the 3-L sequence 8:9:12:16 (C \Bb\ F C). If we put it in the middle we get the 3-L Mousike sequence 6:8:9:12 (C G F C). On the bottom position sits 9:12:16:18 (C G D C). This third possibility became the standard ‘default’ position favored by Greek theorists. Since the traditional monochord had two octaves, one above and one below the mese, the division for the so-called Greater Perfect System gives these frames for the tetrachords: 9:12:16:18:24:32:36 (C G D C G D C). In this framework the tetrachords C G and G D are conjunct on G, while the tetrachords G D and C G are disjunct or separated by a whole-tone (D C). Evidence shows that not everyone followed the rules of the theorists. Alternative frameworks also existed, such as the Lesser Perfect System that puts conjunct tetrachords on the mese. Nevertheless, the bottom position appears to dominate.

Ancient Greek theorists (but not the medieval Arabs) applied the same rule to the tetrachord itself. The smallest interval should sit on the bottom and not in the middle or
on top. Yet the rule seems arbitrary and some evidence shows that it was not always obeyed.

Let’s look at a modest sample of some tetrachords that have historical and/or structural significance. In order to simplify the presentation I’ll give only monochord sequences and not the ‘up’ scale. Also, I will use mainly schisma numbers for interval size rather than the cents numbers. Finally, I will use only the mese tetrachord, the tetrachord built on C as my default tetrachord. This tetrachord has already appeared in our simple 5-L sequence 3:4:5:6 (C G Eb C). Now the 3:4 (498.045c or 254s) increases its numerosity to make families.

One of the most consonant tetrachords is surely the 5-L diatonic 15:16:18:20 (C B A G). It has the large diatonic semitone 15:16 (57s) and the typical combination of just intonation whole tones in 8:9 (104s) and 9:10 (93s). The progressive theorist Eratosthenes must have used this division as a basis for his 19-L chromatic 15:18:19:20 (C A \Ab- G). By taking the AM of the 9:10 he has generated two medial semitones (48- and 45+s). We can rearrange the original diatonic tetrachord to put the semitone on the bottom. Then we get Didymus’ 5-L diatonic 24:27:30:32 (C \Bb \Ab G), intervals 104+93+57. Didymus based his 31-L enharmonic on this division. It goes 24:30:31:32 with its large quarten tones of 29 and 28s. With the 5-L diatonic 27:30:32:36 (C Bb /A G) we see another rearrangement, with intervals 93+57+104. Ptolemy’s Diatonic Syntonon sequence 36:40:45:48 (C Bb Ab G) with intervals 93+104+57 is very close to that of Didymus. The whole tones have been reversed. Another rearrangement makes the 5-L diatonic 120:135:144:160 (C \Bb A G), intervals 104+57+93. Finally, we have the sequence 135:144:160:180 (C B /A G), intervals 57+93+104.

I have now given all six permutations of the combination of intervals 57s, 93s, and 104s. Whatever trio of intervals makes up a tetrachord, it is always amenable to six permutations. The only exception to this rule applies to divisions that have a symmetrical position. For example, the 5-L diatonic 45:50:54:60 (C Bb A G), intervals 93+68-93. It can permute only into 75:81:90:100 (C \Bb A G), intervals 68+93+93, and an offering of al-Farabi 81:90:100:108 (C Bb /Ab G), intervals 93+93+68.

Another 5-L symmetrical tetrachord of enormous and widespread popularity is sequence 60:64:75:80 (C B Ab G), intervals 57+140+57. I won’t give its permutations here. Another historically and structurally significant division related to this one is Didymus’ 5-L chromatic 60:72:75:80 (C A Ab G), intervals 161+36-57. Didymus has combined the chromatic semitone (36s) and the diatonic semitone (57s) that sum up to the 5-L whole tone (93s).

So far my sample has focused on 5-L tetrachords. Here are a few 3-L tetrachords of interest. They naturally require larger numbers. Philolaus’ 3-L diatonic sequence 192:216:243:256 (C \Bb \Ab G), intervals 104+104+46 is also cited by Plato (Timaeus), Euclid (Seventy Canonic), Ptolemy who calls it the Diatonic Ditoniaion, Eratosthenes and many others. It has great historical importance. The permutations also have features. The sequence 216 243:256:288 (C \Bb /A G), intervals 104+46+104 presents the symmetrical
position. Then 243:256:288:324 (C /B /A G), intervals 46+104+104 completes this set. Finally, I offer Philolaus’ chromatic 1728:2048:2187:2304 (C /A \Ab G), intervals 150+58+46, with its very uncharacteristically large semitones. He must have had problems making a small semitone in the 3-L. Such harmony is essentially only comfortable within the diatonic genus.

We have not yet seen an example of a 5-L enharmonic tetrachord. Take the sequence 96:120:125:128 (C Ab Abb G), intervals 197+36+21. Here the chromatic semitone (36s) and diesis (21s) add up to a diatonic semitone (57s). The interval of 197 defines the 4.5 just major third.

I would be remiss not to include some examples of 7-L tetrachords. Perhaps the most consonant form was cited by al-Farabi, the sequence 12:14:15:16 (C Bbb Ab G), intervals 136+61+57. He uses two large diatonic semitones. The tetrachord 18:20:21:24 (C Bb Bbb G), intervals 93+43+118, includes the medial semitone (43s) and the 7-L stretched wholetone 7:8 (118s, which equals 104+10+4). The tetrachord 21:24:27:28 (C A\# G\#+ G), intervals 118+104+32, employs the 7-L chromatic semitone 27:28 (32s). Ptolemy’s Diatonic Malakon 63:72:80:84 (C A\# \Ab- G), intervals 118+93+43 deserves mention. So does Archytas’ 7-L enharmonic 84:105:108:112 (C Ab G\#- G), intervals 197+25+32. Many more 7-L harmonics could be cited.

The most famous 11-L tetrachord is undoubtedly Ptolemy’s Diatonic Hemiolon or Equitable Diatonic. It goes 9:10:11:12 (C Bb A- G), intervals 93+84+77. Theorists call it ‘equitable’ because the intervals approach each other in size. Unlike most conventional tetrachords, it seems to stand almost outside the traditional classification, since it has no real semitones. Its character comes into fine relief when we look at its tempered version in Aristoxenean parts: 11+10+9p, or 93.5+85+76.5s. If the numbers don’t seem to add up here, remember that the tempered 3:4 is schismatic (255s) while Ptolemy’s version adds up to the just 3:4 (254s). The Hemiolon has had a lot of bad press by western writers, who often describe it as ‘ugly’ or ‘only theoretical’. On the contrary, it has a unique charm and it has also been much used. Many variants can be repeatedly heard in Arabic music, usually described as a wholetone plus two three-quarter tones.

The Hemiolon brings to mind a more truly theoretical tetrachord that divides the 3:4 evenly. Al-Farabi gave the tempered version in parts, 10+10+10p (85+85+85s) that sums up to the fourth 30p (255s). If we apply a similar cube-root operation to the just 3:4 (254s) we get an irrational division in schismas 84.66+84.66+84.66. With the slightest tweaking it can become the symmetrical 85+84+85. Using the language of Aristoxenus, it is a scale of five-sixth tones. The tetrachord can also be closely approximated in arithmetical ratios by using (for example) the 109-L sequence 90:99:109:120, in intervals very close to 84+85+85s.

The GM of the just 3:4 (254s) yields the odd interval of 127s, a version of a tone plus a quarter-tone. Foreign to European scales, it can be found in some Arabic scales and in Aristoxenus. As always, it sits half way between the HM (136s) and the AM (118s), derived from the sequence 6:7:8 (136+118s).
I offer just one 13-L tetrachord as a beautiful teaser. It was cited by Avicenna, the sequence 21:24:26:28 (C A#/Ab- G), in intervals 118+71-465+8. This example illustrates that 13-L harmony (like the 11-L) poses difficulties in notation. I developed an approach that uses scala-shifting as a basis, but it need not be explained here. Those interested should consult my paper on the tetrachord: Homage to Prokofyev: A Systematic Generation of n-Limit Just Intonation Tetrachords. There one can find many more historical and alternative tetrachords.

In the set of diagrams I have also applied myself to the inventory of tetrachords for some tempered systems. Here we see that 12-et has only six varieties, 19-et twenty-one, 31-et sixty-six, and 53-et one hundred and fifty-three. The lesson here is simply that tempered systems approaching just intonation possess an enormous variety. Yet this wealth is dwarfed by the potential inventory for n-Limit just intonation. The variety of tetrachords rivals that of octave scales. This table also says that the emphasis on the tetrachord is perhaps less necessary or inevitable in 12-et where the stock is so meager.

PLANETS

The science of Harmonics (or Canonics, to use the ancient term) can be closely tied to the study of Calendrics. They both use the same sort of arithmetic. They both involve the measurement of a time period for a recurrent vibratory event. Something mechanically vibrates in a cycle. The difference between Harmonics and Calendrics only amounts to scale. Harmonics as a discipline deals with very fast vibrations, hundreds or thousands per second. As a result frequency (cycles per second) becomes an appropriate descriptive device. In the case of Calendrics we have extremely long time periods such as 365 days (the earth's orbital cycle). Now we see extremely low frequencies, and it thus becomes more convenient to express measurement by period. But we can always convert the 365 day period into the number of seconds, then invert the ratio to give the frequency. The central problem in Calendrics consists of finding some least common multiple that will integrate disparate cycles, say, the sun and the moon. The task is essentially musical.

Harmonics and Calendrics both qualify as likely the oldest exact sciences to have evolved in history. In my assessment, Harmonics has priority, since it only needs simple numbers for practical results. Calendrics requires what I call 'messy' numbers, complexity. Historically, astronomy started out with musical preconceptions, but over long periods of time it emancipated itself from music and developed more appropriate models for its own particular milieu. Nevertheless, throughout antiquity ties still exist between music and astronomy. Moreover, the further back we look in history, the more one encounters calendrical numbers trying to conform to musical pretenders.

Sumeria-Babylonia offers an excellent example. Here the lunar number of 30 days relates to the solar year number of 360 days in twelve lunations. They use neat musical numbers as their 'ideal' model, but they also recognized that the actuality is a bit more complicated. They managed to make the 365 days more musical by dividing the 360 days into 72 'weeks' of 5 days each. This is their civic year of 360 days or 720 'days and
nights.' The left-over 5 days then forms an extra week ceremonially outside time, acting as the special connector period before the New Year. In this way the year is musicalized by employing the calendrical comma of ratio 365/360 or 73/72 (23 879c), the comma adjustment one makes to bring the musical year back together. Of course, this 365 day year is itself not so accurate, it does better at 365.2422 days. In our modern (Julian) calendar we use 365.25 days (through the device of a leap year) and then make some fine adjustment once a century or so. A calendar always presents some compromise between accuracy and practicality.

The lunar cycle also had to be refined. It sits not at the musical 30 days but rather 29.53059 days, that is, 29 days, 12 hours, 44 minutes, and 3 seconds. In this (synodic) period we move from new moon to new moon. A widespread ancient practice used 29.5 days. Months were alternatively 29 and 30 days (called hollow and full). The lunar calendar remained prominent because it is both short and visible, hence quite practical.

Using alternative months of 29 and 30 days, then 12 lunations make 354 days, over 11 days short of a solar year. More accurately, there are 12.368 lunations a year. How do we reconcile the lunar and solar year? Our modern approach is to adopt a narrowly solar calendar, largely abandoning the moon, and we let the lunar phases 'freewheel.' We use months of different lengths no longer directly tied to the lunar phases. On the other hand the ancients valued the lunar perspective. So they used a lunisolar year of 12 months (354 days) and then sometimes interpolated an intercalary year of 13 months (384 days). It can be done in various ways, but the most successful is a 19 year cycle that gives 12 years of 12 months each and 7 intercalary years. The sequence of 19 years goes: **|**|**|**|**|**|**|**|**|**|**|**|**|**|**|**|**|**. It works because of a least common multiple where 19 solar years (6939.6 days) practically equals 235 lunar months (6939.69 days). At the beginning of the twentieth year the new moons fall on the same calendar date as the civic (solar) year. This one is very good, but even here a slight discrepancy still can be found.

The point in this exercise is to show that the cycles of the sun and moon relate to each other in a complex manner that can't be reduced to a simple musical ratio. We also see this complexity with the planets. Hence the establishment of a Great Year or grand cycle that integrates the whole planetary dance is not a trivial task.

I wanted some idea of the character of this grand harmony. In the early 1980's I discovered a way forward. An octave is a vibrational double. Due to the principle of octave invariance we give a higher octave the same pitch name. Thus one can use powers of two in order to translate a slow vibration into the hearing range, there to assign it a frequency rate and pitch name. At the time I called the approach 'the Octave Law.'

Here is an example of the process, employing the year cycle. I found as accurate a measure as I could find---365.24219879 days. The 33rd octave of this period comes to 272.204 Hz. Given that the standard C on the piano sits at 261.63 Hz, then the year cycle sits around a C# (about 68.6c above C). Again, the moon cycle at 30 octaves yields 420.84 Hz, around an Ab+ (822.9c above C). All the planets have their own values, as well as other natural cycles such as the earth's spin rate (or day cycle). I won't give the
details here—see my paper: *The Law of the Octave and natural resonances*. Here I only point out that the results make a pattern of utter complexity and no simple scale or familiar harmony. In the case of the sun and moon, the moon sits a deformed fifth from the sun (753.3, about a quarter-tone sharp). We get similarly complex positions with the other planets. The harmony is strange indeed.

The most intriguing result of this series of calculations was the observation of a ‘clumping’ of values around a G pitch. The standard piano G sits at 392.00 Hz. The sunspot cycle (about 11.1 years) comes to 392.37 Hz. The sidereal day (daily rotation of the firmament) comes to 389.425 Hz. The long solar cycle of 179 years harmonizes the planetary orbits with relation to the displacement of mass in our solar system. It comes to 389.30 Hz. The solar day (the period between the successive culminations of the sun) sits at 388.36 Hz. These values all sit between G (392.00 Hz) and \( \overline{G} \) (387.33 Hz). Also, many planets have complex harmonic relations with the two most obvious solar cycles (sunspot and long cycle). I have concluded that G is the best candidate for the predominant natural frequency in our solar system.

Any ancient tuner that tried his hand at such investigations would quickly have become frustrated. What they wanted was a simple harmony, preferably one in actual use. At any rate we have no ancient evidence of such an investigation. Instead, all the ancient commentators on the ‘cosmic harmony’ focussed on the distances between the planets, not on their temporal periods. They imagined the cosmos as a series of concentric circles or spheres around the earth in the center. They had no means of actually measuring the distances between the spheres. Thus they felt free to indulge in their imagination and input whatever scale that they considered appropriate.

This model associates itself closely with a monochord, earth at the mese position, the outer stellar sphere on the open string an octave away. Controversy arises over which scale to use. Many writers gave alternative suggestions or hints, most omitting numbers entirely. In the diagrams I have presented one of the more likely candidates, the double-strand (5-L) Diatonic Ogdoad ruled by the double 72:144. Another candidate suggested by the diagram is the 3-L comma-shifted ‘cousin’ of the 5-L scale. It was given by Euclid and ruled by the double 1152:2304. Of course, these two don’t exhaust the suggestions. I have collected the ancient evidence together with commentary in my paper: *The Cosmological Monochord*.

Perhaps the greatest single ancient Great Year or grand cycle number is the precession number 25,920 years. The longest astronomical cycle to be visible to the naked eye, it amounts to one degree of arc every 72 years. The more accurate number is slightly smaller, but no matter, the ancient value had impeccable musical pedigree. It equals 360 times 72, also 432 times 60, and yet more musical factors. The Sumerians had a master number or ‘holy’ number that integrates all of their cosmic cycles. It is 12,960,000 which is 25,920 times 500. The number 12,960 makes half of 25,920. Many musical numbers can be included in this family. All such numbers are 5-L and can be mapped on the matrix.
OCTAVE EQUIVALENCE

The octave is special due to its unique relation to unity. The 'other' and the 'one' reflect each other. Hence we give the octave the same pitch name, experiencing it as a sort-of higher reflection. The octave had a major impact on early Greek philosophy, where it underpins the isomorphic relation between the microcosm and the macrocosm. It also sits under the philosophical notion of the unity of opposites. The ancients modelled philosophy the same way that they modelled music; that is, they recognized two aspects that are always present—changing and unchanging. The changing element comes from the ongoing measurement of time and number. The unchanging element comes from the ubiquitous movement within a cycle, from cyclicity or periodicity. The octave underpins this ancient perspective. It stands for cyclical identity.

I have assumed octave equivalence in my two mapping schemes—the triaxial matrix and the circular graph. I also employed it for the 'octave law.' But in this section I want to remind the reader that octave equivalence is really just a useful fiction. When intervals are displaced by an octave they don't maintain the same status. Some lose in consonance, others gain.

The octave itself seems to lose little at the double octave 4/1, but at the triple octave 8/1 it already comes into border territory. At four octaves 16/1 it becomes decidedly dissonant. Such distant octaves as I have used in the 'octave law' procedures have very little actual musical meaning. They serve more as analogies.

The musical fifth 3/2 gains consonance when it becomes the twelfth 3/1. Thereafter it loses as 6/1 but it is still consonant. With 12/1 the consonance disappears. Unlike the fifth, the fourth 4/3 loses when octave displaced as the eleventh 8/3. Now it sits in the border zone. At various points in history theorists have argued whether 8/3 is consonant or not. But there's no argument with 16/3.

The just major third 5/4 gains as 5/2 and yet again as 5/1. But the just major sixth 5/3 loses heavily as 10/3. Thus we can say that the sixth is somehow a more fragile interval than the third. The third has more stability.

The just minor third 6/5 loses much when it becomes 12/5. It becomes 'rough.' The septimal minor third (or augmented second) 7/6 embodies less consonance than 6/5, but its octave 7/3 gains over 7/6. Oddly enough, the 7/3 sits superior to the 12/5. Here the superiority of the 5-L minor third over the 7-L version is revered when we use an octave displacement.

The just minor sixth 8/5 already sits on the borderline, and loses consonance as 16/5. This borderline status in the 5-L minor third and minor sixth results in the widespread judgement that the intervals form 'imperfect' consonances. In my opinion the septimal tritone 7/5 also falls into this class. The only septimal interval that gains on octave displacement is the 7-L minor seventh (or augmented sixth) 7/4. It becomes 7/2 and 7/1. Historically, theorists have argued over whether it is consonant or dissonant. In
my judgement it seems a form or consonance, but musically it has always been used as a dissonance, in the context of seventh chords. Thus its status is controversial.

The soft dissonance of the 3-L whole tone 9/8 gains as 9/4, while its 5-L cousin 10/9 loses as 20/9. The ‘stretched’ septimal whole tone 8/7 epitomizes the borderline, and loses as 16/7. I could go on, but instead only note that dissonant intervals also adjust their status with octave displacement. Thus the theoretical tool of octave equivalence has its limitations.

This situation isn’t confined to intervals. Take the simple just major triad chord. As a three-note chord in its various inversions or voicings, it forms components of the Harmonic Series (on C). The most harmonious position is C G E, harmonics 2 3 5. Then comes 3:4:5 (G C E) and 4:5:6 (C E G). These are decidedly better than 3:5:8 (G E C) or 5:6:8 (E G C). The worst in this lot shows 5:8:12 (E C G). The same kind of hierarchy in consonance appears in the minor chord (here C minor). As components of the Harmonic Series all of them are dissonant. The best is 6:10:15 (Eb C G), then 10:12:15 (C Eb G). The status deteriorates as 12:15:20 (Eb G C), 10:15:24 (C G Eb), and 15:20:24 (G C Eb). The worst sits at 15:24:40 (G Eb C). In short, the arrangements of notes in a given chord also affect the status of a harmony. An octave equivalent model such as the matrix assigns a chord type by the pebble pattern, but it remains a rather abstract device that hides the reality of actual chord voicings. Nevertheless, the abstraction is a useful means to get an overview.

The octave has defined the cyclical boundary for all of the tunings given thus far. However, it need not necessarily be the case. We could use as the basis for a repeating pattern the twelfth 3/1 instead of the 2/1. Then one can generate a whole new family of tunings in which the octave is not necessarily just. Such a family includes new equal temperaments and irregular temperaments. The potential resources are vast. They have only been explored recently, especially on the clarinet that overblows on the twelfth. Such tunings have no historical precedent before the 20th century. With modern electronic media new possibilities have opened up, since computers excel in number crunching.

TOLERANCE

How far can a musical interval be mistuned and yet remain recognizable and usable? The question usually arises within the context of temperament, often as a justification or an evaluation. Historically, we first see the issue openly broached in Aristoxenus. For him, as well as the other classical theorists, only the 3-L ratios of the fourth, fifth, octave, and double octave define the consonances. 5-L ratios such as 5/4 are classed as discordant, although ‘near’ consonant. Such intervals are more malleable or mutable than the consonances. They make themselves more open to manipulation. Aristoxenus argued in support of 72-et that the consonances are very good, and the dissonances have a variety of possibilities. In this regard he considered it permissible to simulate or represent a continuum of monochord positions. He wanted to map the whole territory. His argument seems entirely reasonable to me, although historically it has been repeatedly attacked.
Aristoxenus never claimed to have initiated the subject of tolerance. Among predecessors, he referred to a statement from the lost book by Lassus of Hermione; indeed, it makes our only surviving piece of information from his book. Lassus, a contemporary of Pythagoras in the latter sixth century, said that notes have breadth in the scale; that is, when they are plotted on the monochord they make finite portions of the line rather than mere points on it. This observation does not seem so unusual to me, since the fret position always needs a little adjustment anyway. It has a little lea way or ‘smear.’ Lassus’ statement could well have come out of practical monochord experience. At any rate we have here the earliest surviving potential opinion on the thorny subject of tolerance.

It appears that Lassus was a progressive tuner. Nor was he the first. His teacher Epigonis of Sicyon built a multi-stringed experimental instrument tuned in enharmonics and intended for acoustical research. According to Aristoxenus he founded a school of harmonists. Plato (a cryptic conservative) probably referred to this school in the Republic when he spoke disparagingly (or satirically) of men who ‘torture strings by stretching them on racks,’ listening attentively, trying to find the smallest possible practical interval by which to measure the other intervals. Thus, in the second half of the sixth century we already see evidence of polemical dispute between progressives and conservatives.

Circumstantial evidence indicates that the Milesians were also progressive. Pythagoras represents a conservative reaction to Milesian radicalism. Nor was the Pythagorean Society monolithic. It soon, perhaps already during the lifetime of Pythagoras, split into two camps, conservative and progressive. The conservative camp, called aconsmatici (hearers), just copied the ideas of Pythagoras. The progressive camp, called mathematici (scientists), conducted research led by Hippasus. Apparently, Hippasus was ostracized by the society for revealing information about irrationals—a taboo subject for Pythagoreans. The issue of tolerance must have surfaced in these polemics.

The famous movement of early Greek philosophy displays a mixture of conservatives, centrists (e.g., Heraclitus, Empedocles) and progressives. Among progressives two names stand out. Anaxagoras supported an infinite number of elements (i.e. the n-Limit). He represents the guiding light for the progressive school of Ptolemy. Anaxagoras also has the distinction of being the only early writer I can find to shift back and forth, feeling at home both in the arithmetical and the geometrical sphere. He notably used the language of ‘parts’ and Aristoxenus likely took the terminology from him. Special mention should also be made of Democritus, who apparently loved irrationals and wrote a book about them, unfortunately lost. Temperament tends to be atomistic in its orientation. In this milieu tolerance must have been a hot topic. Aristoxenus’ progressivism didn’t come out of the blue.

While defending his description of the tetrachord fourth in terms of ‘parts’ rather than traditional ratios, Aristoxenus says that the ear can more easily recognize and define concords than discords. This is so because concords have either ‘no locus of variation’ (i.e. the octave), or else they have ‘an inappreciable locus,’ (e.g. the schisma-shifted fifth). On the other hand, discords have a much wider variability and hence have less
'definiteness.' Consequently, 'the ear is much more assured of the magnitude of the concords than the discords.' This lack of definition in the discords makes them amenable for description as a continuum of 'shades' between the consonances.

Aristoxenus fixed the consonances and let the others be malleable. Underneath this practice sits a certain attitude to ratios. Put in another way, the more consonant the ratio the narrower the acceptable zone of alteration. Dissonant ratios have lots of 'play.'

These observations of tolerance show remarkable insight and reasonable judgement. Most theorists tend to agree with this stand. The controversy mainly arises over the degree of variability—which intervals can be detuned and how far. Everyone agrees that the octave cannot be tempered at all. For the other intervals the consensus breaks down. I use a practical criterion as a rule of thumb. If various music cultures have used a given level of alteration over long periods of time, then I deem this degree of alteration acceptable or passable.

In the example given earlier, the fifth 3/2 can be altered up to around 3s, almost a clisma. Much practical historical use of various meantone temperaments and well temperaments confirms this boundary. In theory the fourth 4/3 should have a little wider berth, but in practice it has also sat around 3s.

The major third 5/4 naturally has greater variability than the fifth. The modern practice of 12-et makes it 7s sharp. Some complain that it is not very good, but it can also be said that it is not really bad either. Medieval Europe, tuning 3-L, accepted thirds a whole comma (11s, almost three clismas) sharp. They correctly labelled them dissonant, but continued to happily use them for hundreds of years. The same can be said for the historical Chinese tuning culture. Moreover, well temperaments have accepted variability in the thirds and sixths between about 3s and 12s. I conclude that the medial consonances have around 12s or three clismas as their limit of acceptable alteration.

Soft dissonances like the 9/8 have an even wider space of variation. The standard meantone tuning makes it 5.5s flat, to good effect. The comma-lowered 10/9 also sounds like a wholetone. One can go even a little lower, until we reach a 'crisis' point by the three-quarter tone. On the other side, the stretched wholetone 8/7 seems to sit close to the boundary, which lies a little beyond it in a 'no man zone' about a comma lower than the 7/6. This boundary is the interval that we sometimes call a wholetone plus a quarter-tone. This gives the variability of wholetones around three commas.

Of all the dissonant intervals, semitones have the most variability. What we call a semitone can vary from the small chromatic semitone of three commas all the way to the six-comma stretched semitone. At the lower end the boundary between a semitone and quarter-tone is not distinct, though an interval of two commas is definitely called an enharmonic. At the wide end beyond six commas it approaches the three-quarter tone. Between the quarter-tone and three-quarter tone sits a variability of around four commas. The 12-et semitone of 51s sits around the middle of this spectrum—not too big, not too small. For this reason, some argue, it makes a good representative. Since most
instruments have flexibility anyway, we can always squeeze it here or stretch it there according to the context and our inclinations. This wide malleability of semitones serves the aim of greater expression.

This analysis of tolerance doesn’t seem very controversial. However, I sometimes see a counter-argument against the above analysis, a criticism that cannot be ignored.

Perhaps we should enforce a narrower band of tolerance around semitones so that a given semitone can always be distinguished as chromatic or diatonic. This argument hangs on the desire for an enharmonic resolution of harmony. Obviously, given a deeper resolution more distinctions must be clearly defined. In this case tolerance must be adjusted in order to preserve the characteristics of this resolution.

Similar arguments can be made for other dissonances, and extended into a critique of the whole notion that dissonant intervals necessarily have a wide variability. This argument appeals to the supporter of Ptolemy’s n-Limit arithmetical approach. Take the three-quarter tone just ratio 12/11 (77s). One can argue that it has only a narrow critical band. Not far below it sits the stretched semitone of 68s, which is still heard as a semitone. In my experience I don’t hear the transition till around 72s. Moreover, the band extends only to about 11/10 (84s), where I start to hear it as a squeezed whole tone. Arguably the transformation is in place by 87s, well before the 5-L whole tone 10/9 (93s). In effect, the movement from a semitone to a whole tone acts like a wave motion that crests at a critical point. Hence a ratio like 12/11 should be tuned as exactly as possible, avoiding alteration.

This argument assumes that we desire a specific and rather esoteric interval—the three-quarter tone. Since it sits between such common intervals as the semitone and whole tone, and we are conditioned through habit to hear semitones and whole tones, the definition of the three-quarter tone must be quite specific and narrow. Nevertheless, it can still be represented by a tempered surrogate, for example Aristoxenus’ three-quarter tone (76.5s) or the 31-et version (79s).

It all comes down to attitude and resolution. If we just want a ‘generic’ diatonic semitone, then instead of 16/15 (57s) or 15/14 (61s) we can be satisfied with a tempered replacement like Aristoxenus’ version of 59.5s or the 31-et version at 59s. The just ratios are themselves only a clisma apart, and the clisma is conveniently divided so that the error in the tempered version is at most 2s—not much. On the other hand, if we desire a proper clisma resolution and demand distinctive 5-L and 7-L diatonic semitones, then the band width must be eliminated or further refined in an appropriate manner.

Hence the value of tolerance as a concept depends upon aesthetic considerations. A follower of Ptolemy’s school prefers to bring out the fine characteristics of the ratios, including their peculiarities and complex dissonances. Given this aesthetic, the decision must lead to an avoidance of any temperament. Alteration is anathema. Instead, one endeavors to tune every arithmetical ratio so exact as possible. Here we see two conflicting schools in tuning philosophy. Both approaches have a long history that still is not resolved.
PHILOSOPHICAL FOUNDATIONS

At this point I've given a fair amount of technical information about tuning systems. It's necessary in order to get a feel for the territory. But I've also tried to make it easier on the reader by restricting my comments mostly to semitones and other small intervals. I've said nothing, for example, about the morphology of tritones. Ultimately the technical details are less important than the overall picture—one of abundant diversity.

Even with a restriction to meantone temperaments we have a good choice. The rubric for meantone temperaments can be tweaked in subtle ways to generate a large family of useable systems. The same can be said for well temperaments, where the contrast between the common keys and the distant keys may be deepened or made more shallow, allowing us to sculpt customized versions of the ideal. Moreover, there's no shortage of interesting equal temperaments with fascinating features. I have emphasized schisma, clisma, comma and diesis resolutions for their structural importance, but these exemplars don't exhaust the options. Aristoxenus' 72-et gives an example of a good resolution apart from the primary progression. Needless to say, if we admit the subtleties of irregular temperaments the choices prove vast.

The same can be said for just intonation tunings. If we stay 3-L a large inventory still ensues. The move to 5-L harmony opens up numerous further possibilities that only rapidly increase if we proceed to 7-L or 11-L harmony. In the diagrams I stopped calculating ratios when I was working on the 13-L inventory, although I already had sketched up a bit of 17-L material. I decided to leave the unfinished 13-L table as is—it's as good a place to stop as any. The reader need only be aware that we can move up the series of prime numbers as far as we wish. The resources appear practically limitless, but saturation sets in.

Keeping this diversity in mind we can now examine some of the more fundamental principles and issues of the monochord environment. To this end we need to dip into early Greek philosophy once more. Anaxagoras gives more pertinent information for the science of Harmonics than any other single philosopher. He has comments on the topics of opposites, reciprocals, infinite divisibility, the plenum, infinite types, proportionality, mixture and separation, and more. Much of what he says only makes sense in the context of Harmonics. Of course, he is not the only valuable source (re Herachitus, Parmenides, Empedocles), but Anaxagoras amply demonstrates a mastery of all the fundamental issues.

An overview of Anaxagoras is impossible here, but we will look at just three of his statements that apply to the tuning environment. Perhaps his single most famous pronouncement has been called by scholars Universal Mixture—everything is in everything. Another closely associated principle I have called Omni-divisibility—everything comes out of everything. The importance of mixture also informs the principle that I name Integration—every part forms a mixture like the whole. These statements propel us into the heart of the vibratory problematic.
Anaxagoras’ statements presuppose the core musical principle of Wholeness—parts form wholes within wholes or more simply put, parts are wholes. When we divide a whole thing into parts, say chopping an apple into slices, we have eliminated the one whole to produce the many parts. The slices are not to be confused or equated with the whole apple. However, musical materials don’t behave this way. When we take the initial one-whole (the open monochord string) and divide it, say, into the octave 1:2 and the harmony 3:4:5:6 we have generated many, but in doing so we have nevertheless preserved the one, not killed it. The harmony is just a differentiated avatar, or active agent, or emanation of what is already implicit in the one. The open string implies the Harmonic Series. In the case of our monochord division, the octave 1:2 or any other subset of our harmony (say, 4:5 or 3:4:6) forms a part of the whole that is itself a whole. I am using the term ‘parts’ here in an expansive sense, not necessarily restricted to Aristoxenus’ 72 parts to the octave, or even to irrational values in general. In musical harmony, subsets, whether rational or irrational, define parts that are simultaneously wholes within wholes.

Anaxagoras gives us the musical principle of Integration—every part forms a mixture like the whole. Whether we look at the whole sequence or any of its subsets, we see mixtures everywhere. A just intonation harmony involves a mixture of prime numbers and their composites. Anaxagoras and others poetically called them seeds and their growths, using a plant metaphor. The poet Empedocles more transparently called them roots and their compounds. Additional metaphorical names include limbs of the cosmic person, powers of domains, ancestors of the tribes, and parents of the families. Aristotle called them elements, a term never used by the early philosophers themselves. Aristotle recycled early philosophy to suit his own interests. He wanted to take philosophy-science away from its old musical concerns and put it on a new and more biological foundation. To this end he reinterpreted the early philosophers as scientific materialists trying to find the primary substance or original stuff of the physical world. Aristotle’s revisionist view of early philosophy has become an unquestioned orthodoxy, a straitjacket that has made an understanding of Anaxagoras and his compatriots practically impossible. For the intelligible notion of ‘parts and wholes’ shifts radically when we move away from harmony and toward material substances. The early philosophers only become consistent and logical when the musical context is restored.

Anaxagoras confronts us with the more difficult musical principle that I call Omni-divisibility—everything comes out of everything. For any ratio can act as a reference point for further division. Harmonies form subsets and supersets of each other by division. They can be transformed into each other by using the appropriate factor. Thus harmonies have ambiguous paternity. One can arrive at a given harmony by taking different routes that end up at the same place. For example, take the 5-L monochord sequence 12:15:16:18:20:24 (C Ab G F Eb C). We could derive it directly from the octave 1:2 by using the factor 12 to get 12:24. Or we could derive it from 3:4:5:6 by using the factor 4, since 3:4:5:6 equals 12:16:20:24. Again, we could derive the sequence from 4:5:6:8 by using the factor 3, since 4:5:6:8 equals 12:15:18:24. Or we could derive
it from the Mousike sequence 6:8:9:12 by using the factor 2. Harmonies change into each other.

Anaxagoras' most subtle gift is the musical principle of Universal Mixture—everything is in everything. Unfortunately, it breaks down into nonsense when we consider it within Aristotle's materialist paradigm. In fact, it only works within a musical context. Scholars who have been perennially perplexed by Universal Mixture have tried to confine the doctrine to Anaxagoras alone, but to no avail. The notion is actually quite widespread among both poets and philosophers, where it is generally expressed as 'the All is steered through the All.' In Parmenides, the central goddess (the Dyad) 'steers all.' According to various ancient writers, the distinctive and colorful term 'steerage' comes from the great Milesian Anaximander. But the doctrine is most famously stated by Heraclitus, who said: 'Wisdom is the One, to be skilled in true judgement, how all things are steered through all things.'

The notion of the All proves fundamental to early Greek philosophy and to Harmonics. It first surfaces in the Milesians. Using modern terms, the All refers to the totality of both rational and irrational harmonies. The set of real numbers unites the sets of rational and irrational numbers. The Greeks had no descriptive term for real numbers, but they were perceptive enough to recognize that they form a continuum in the division of a line segment. By whatever tuning route we take, eventually we approach saturation. Monochord fret positions were traditionally seen as something like beads on a string. When the string becomes thick with beads they crowd each other out and we approach saturation, the All. Even when we impose the maximal restrictions by using only 3-L ratios in our harmony, the lack of some principle of Limit leads to the Unlimited—infinite divisibility, the central problematic of both early Greek philosophy and the science of Harmonics. How can we best conceive or describe the continuum?

The traditional, likely inherited, way to model harmony I name the digital perspective. It consists of beads on a string, or number sequences like 3 4 5 6. A space, called the 'void' separates the number positions. The traditional model can be summarized as 'numbers and the void.' However, the pioneering Milesians groped for an analog perspective on harmony, where the void is potentially banished. Seeking to integrate irrationals into the fabric, they faced the issue of infinite divisibility. Their model consists of a continuum between what is poetically called 'hot and cold' or 'rare and dense' or various more descriptive metaphors for the polarity in the octave space between 'Heaven and Earth.' That microcosmic space displays a unity of opposites.

The framework is most explicit in Anaximenes, the youngest, most articulated, and most influential of the Milesians. He replaces the traditional 'roots and compounds' with the phase-changes of a single 'meta-element.' He wants an analog or abstract replacement for the old system of arithmetical sequences. Although the evidence is entirely circumstantial, it nevertheless points to incipient temperament and atomism. Much later in time Democritus affirmed 'atoms and the void,' but the early philosophers, most of them, wanted to banish the void altogether. Only conservatives like Pythagoreans defended the void. The greatest of the early philosophers, Heraclitus, took the
paradoxical stand that 'it is both a void and a plenum.' He evidently understood the principle of Universal Mixture.

THE THREE WAYS AND UNIVERSAL MIXTURE

In order to get a grip on the meaning of Universal Mixture, we need to examine it within the context provided by Anaxagoras. According to his fragments and the testimony of Simplicius, Anaxagoras recognized three ways that forms (musical structures, harmonies) can be conceived, or rendered. These three ways have strong associations with three monochord states. Here is a brief explanation, entirely avoiding the Greek technical and philosophical terminology.

In the first way, all structures are gathered together into a Whole. Instead of making this division or that division, make the thought experiment of producing every division at once, both rational and irrational. As a result we generate the plenum in which the void is banished. Now we sit in the analog continuum, the solid line. One could call it the 'All-at-once' or the 'All-together-now.' Since everything is present at once, it sits paradoxically outside of time, change, and motion. Moreover, the All or Whole (every division) results paradoxically in the equivalent of no division (the One). Thus the modality of the first way can be associated with the open monochord string before any division is even made. Here in the All that is also the One we find the natural home of Universal Mixture. Since everything inhabits the same place and time, then 'everything is in everything.' Perhaps the notion was originally intended for this context alone, but we will see that it has a much wider relevance. The first way gives rise to a realm of unchanging atemporalia and its associated philosophy of Being.

In the second way, the Ali is poetically dissociated into a polarity, a unity of opposites, through the capability of the Dyad—the vibratory power of the number two. The monochord bridge is placed in the middle of the string to make the octave sequence 1:2. Here we place the metaphorical house of harmony where the families are generated, the mixing bowl where every recipe is concocted, the playing field where the warring elements (3, 5, 7, 11...n) do battle. The octave forms the essential microcosm within the macrocosm, where the All and Universal Mixture still rule. The second way makes the realm of unchanging temporality, cyclicity or periodicity. It defines the gateway realm between the unchanging and changing spheres. It also acts as the controller of the vortex of further division—the master ratio. Not without reason it has been called the first logos.

In the third way we make further divisions and so enter the world of changing temporality where the power of division, differentiation, mixture and separation takes hold. Here alternative divisions give birth to specific harmonies, both rational and irrational. Now we sit in the realm of name-and-form, of alteration, of varying resolution, of incipient complexity. Here the 'ordinance of Time' rules and gives rise to a philosophy of Becoming. One would think that here at last the notion of Universal Mixture can be exiled from further use. For any given sequence such as 3:4:5:6 clearly distinguished between what is inside the set (the mixture) and what is excluded from the set (also a defining characteristic). Here it appears that everything is not in everything. Rather, we
see an 'in crowd' distinguished from an 'out crowd,' the arithmetic of set divisions. Discrimination and judgement rules the third way.

These three ways exhaust the manner in which forms can be taken. In short, we witness an interpenetrated unchanging and changing sphere, giving rise to simultaneous (and compatible) philosophies of Being and Becoming. The two modalities meet in a gateway realm that communicates between them and contains both. In the third way it appears that Universal Mixture is breached. However, even here we can detect the subtle workings of Universal Mixture in a number of contexts.

One important context involves what I call the musical principle of Psychoacoustical Equivalence. It seems that rational and irrational ratios form entirely separate continents in the world of harmony. Throughout history we witness a conflict or contest over the two contenders. One school favors some form of just intonation, the other some version of temperament. Yet the issue is ultimately a red-herring. Once we reach a certain level of complexity in either camp the other camp can be well simulated. The psychoacoustical effect converges—we hear the same thing. In the example I have given earlier, the irrational ratio of the square root of two (600c) can be closely simulated by the 11-L just ratio 99/70 (600.088c). Any irrational ratio can be replaced by a just version that is an acoustical equivalent as long as we allow the resource of n-Limit ratios. Coming from the other side, any rational ratio norm, such as 5/4, can be successfully simulated by using some irrational approximation that is really close in size. Hence the two continents of harmony, rational and irrational, interpenetrate each other in a strange loop (in the continuum) that displays a form of Universal Mixture.

Another context very close to this one concerns the issue of tolerance—the controlled detuning of just ratio norms. In my earlier discussion of tolerance I naturally associated it with temperament, but that need not always be the case. One can also detune using a just ratio. For example, the norm 3/2 (701.955c) could be replaced by the 11-L ratio 220/147 (698.0c). It sounds like a typical well-temperament or meantone temperament irrational, related to the standard meantone ratio (606.85c). The point here is that the psychoacoustical experience of detuning proves to be independent of the whole issue of rationals versus irrationals. Here we see another form of Universal Mixture that I call Universal Tolerance.

Another example of the workings of Universal Mixture concerns the functional equivalence of disparate system models or rubrics. We have already seen the practical identity of quarter-comma meantone tuning (a tempering of the open line of fifths) and 31-et (a multiple division of the octave). Although the two systems come from completely different design criteria, they end up being sonically equivalent. This sort of thing occurs over and over with different ratio structures. We see here another form of 'everything in everything' that I call Inter-modelability.

Yet another related form of Universal Mixture concerns the ability of various harmonies to nest within each other or form variants of the same thing and each other. For example, take the multiple divisions 12-et, 31-et and 53-et. Each of these resolutions
(semitone, diesis, comma) exhibits a different character with its own peculiar features unique to it. The differences in resolution can be conceived as somewhat like the depth of field in a camera. A deeper resolution brings out more detail that has been tempered out of the shallower resolution. Though they differ from each other, yet all three exemplar resolutions form variants of the same thing. In all of them we can find (for instance) a C major chord and scale. In each case it is slightly modified in a different way according to the local rule of temperament. In spite of the differences they are identifiably the same, a C major chord and scale. Each system puts the compromises in uniquely different places, but they prove to be compromises of the same thing and each other. Here we have yet another way in which ‘everything is in everything.’

The notion of Universal Mixture applies to harmonic materials in a peculiarly intense manner. They show this characteristic because they constitute wholes within wholes. I continue with two more important contexts for Universal Mixture.

INTER-DIMENSIONALITY

We naturally organize arithmetical ratios through the key concept of dimensions: 3-L, 5-L, 7-L, 11-L and so on. The prime number series rules. Each dimension seems to be quite distinct from the others. Each level has its own unique and characteristic ratios, and its own ethos or personality. Yet these nominally disparate worlds bleed into each other and co-habit a common psychoacoustical zone, in a manner akin to the intertwined relation between rationals and irrationals. One sees here yet another context for the notion of Universal Mixture. ‘Everything is in everything’ in a musical manner that I now call inter-dimensionalality. One sees this property within the relations between 3-L and 5-L harmony, between 5-L and 7-L harmony, and so on among the prime numbers. Even though 3-L and 5-L harmony proves to be theoretically and mathematically distinct, they entangle each other in a strange loop.

3-L harmony, the line of fifths, is known for its excellent fifths and fourths, and its dissonant (comma-shifted) thirds and sixths. 5-L harmony, the surface or triangular matrix, corrects the thirds and sixths. But it also makes it clear that every interval can be comma shifted, even the fifth. One already sees a dissonant fifth in the 24:48 Diatonic Heptad scale, whose ‘up’ pitches are C D E F G A B C. The fifth between D and A is a comma-lowered version of ratio 40/27 (680.449e or 347s). 5-L harmony precipitates a comma resolution of harmony. However, this comma-based perspective can largely be ignored within 3-L harmony, since the comma does not appear until a long line of fifths (twelve) has been produced. Practical scales have just five to eight members, so the comma need not be an issue. However, the fact that a 3-L comma appears at all points to subtle relations between 3-L and 5-L harmony.

If one wants to maintain the 3-L character of the harmony, with its dissonant major third, the expansion of the line must be strictly limited to seven operations, generating the important structural harmony called the Diatonic Ogdoad or ‘full’ diatonic. As soon as we add a ninth member we cross a boundary and encounter a quasi 5-L major third. This pitch is a schisma-lowered version of the 5-L norm. Since the
schisma is practically subliminal, this 3-L impersonator sounds like the real thing. In short, the schisma interval acts as a connector between 3-L and 5-L harmony, enabling perceptual 5-L characteristics within a 3-L theoretical structure. The schisma shift supports the inter-dimensionality between 3-L and 5-L harmony.

One can see these relations most clearly on the Matrix diagram. Starting on the generator C, tune along the line of fifths in a westerly direction (the monochord orientation). At the ninth member one sees E, the schisma-shifted version of the 5-L norm E. In order to avoid this 5-L intruder, we must restrict the line of fifths from C to B, an ogdoad. This 3-L Diatonic Ogdoad has eight possible modes. The most consonant mode, with the smallest monochord sequence numbers, sits on the line of fifths between D and \(\semitone{\flat}{Bb} \). This important harmony 'competes' with the more consonant 5-L cousin that makes the comma shift, where \(\semitone{Bb} \semitone{\flat}{Bb} \semitone{\flat}{Ab} \semitone{\flat}{Db} \) becomes \(Bb \semitone{Bb} \semitone{Ab} \semitone{Db} \). Now we have the 'double-strand' 5-L Diatonic Ogdoad. Both the 3-L and 5-L cousins are extremely widespread in the traditional tuning cultures. Of course, other modes also have use. The 3-L ogdoad mode from \(\semitone{Bb} \semitone{\flat}{E} \) to \(\semitone{Bb} \semitone{\flat}{B} \) sits symmetrical to the harmonic axis (G C). The mode bequeathed to the Middle Ages by the late Roman Boethius employs the D to \(\semitone{\flat}{Db} \) mode, but using A instead of my chosen C as the move or generator tone. In my notation, which is functional (i.e. showing relation to the tonic and the existence of regional boundaries), his pitches run from \(\semitone{\flat}{Bb} \) to \(\semitone{\flat}{Bb} \). Of course, Boethius' notation is not functional in intent and makes no concessions to the very possibility of 5-L norms. Hence his line of fifths becomes simply B to Bb.

The 3-L Diatonic Ogdoad norm persisted as a European tuning reference for many hundreds of years. During the fourteenth century, a time of cultural extremism, the line of fifths expanded to twelve members (the Dodecad), breaching the ogdoad boundary, and paving the way for the eventual acceptance of the 5-L norm. For the 3-L Dodecad contains a full simulated double-strand ogdoad in its set. On the Matrix, extend the line of fifths from \(D\) past \(\semitone{\flat}{Db} \) out to A. The set is now complete. Now both of the diatonic tuning paradigms (single and double strand) are found in both dimensions, 3-L and 5-L. The attenuation of these two paradigms into each other creates the play ground of well temperament. When we tune the long 3-L dodecad from \(D\) to A even the dissonant comma-lowered fifth is reconstituted. Compare the fifth D A (680.449c or 347s) and the fifth D A (678.50c or 346s). This integration between the single strand and double strand diatonic paradigms extends into the triple strand chromatic frame. The triple-strand dodecad has relations with both the double strand and the single strand. They form comma and schisma shifted versions of each other.

These relations illustrate the subtle ways in which 3-L and 5-L harmony interpenetrate each other, the workings of inter-dimensionality.

A similar case can be made for the relation between 5-L and 7-L harmony. The 7-L is strictly speaking the solid dimension or three-dimensional web. See the diagram called: a three-dimensional tuning matrix for 7-L harmony. However, it can also be mapped over the triaxial matrix through a close substitution. See the diagram called: the feasibility of mapping 7-L ratios. This time the attendant microton is the 7-L clisma.
225/224 (7.712c or 4s). It converts the 5-L pitch A♯ of ratio 225/128 (976.537c or 498s) into the 7-L A♯ ratio 7/4 (968.826c or 494s). Similarly, the 5-L pitch F♯ of ratio 45/32 (590.224c or 301s) becomes the ratio 7/5 (582.512c or 297s) and the 5-L D♯ ratio 75/64 becomes 7/6. The whole tetrad of pitches A♯, D♯, G♯, and C♯ are amenable to clisma shifted substitutions. Moreover, one can locate an analogous reciprocal location centered on the major triad ♭b♭, ♭b♭, ♭b♭ and added ♭♭ and C♭. For example, ♭♭♭♭b♭ of ratio 256/225 (223.461c or 114s) becomes 8/7 (231.174c or 118s). In this region the clisma is added rather than subtracted. These two 7-L zones within the 5-L fabric generate autonomous tri-axial matrices that cover many useful 7-L ratios. Deep immersion in 7-L harmony pushes one toward a clismatic perspective on structure. Everything can be clismatically shifted in various ways within a clisma resolution.

The clisma is an odd and challenging interval, no longer subliminal but still quite minimal, since it takes about three clismas to make a comma. Working with the 5-L dissonances I found them to be passable or acceptable (depending on the timbre) as surrogates for the 7-L norms. The same goes for the 53-et versions. The clisma is also an interesting interval in that it has a prominent 5-L version nearly in the ratio 15625/15552 (8.107c or 4s). The 612-et multiple division tempers away the difference between the two clismas, offering the compromise 4s. Wherever we look in tuning systems, compromises are always being made between competing ratio candidates. Even the schisma is bound to a micro adjustment. The 5-L ratio 32805/32768 (1.954c) becomes in 612-et 1.960c. The clisma is very much bound up with the schisma in the fabric of harmony. Moreover, the halfway point between them, the semi-clisma or double-shisma, also has prominence. All of these subtle relations illustrate the interpenetration of 5-L and 7-L harmony—their inter-dimensionality.

With 11-L harmony we move into a world in which the principal ratios form exotic intervals like the neutral third and neutral sixth. Such harmonies have not been universally accepted by different music cultures. In my studies of Indian tuning traditions I saw very little evidence of even the 7-L, let alone higher orders. The Indians appear to have perpetuated the ancient 5-L mainstream. On the other hand, in the Middle East and surrounding areas (North Africa, Balkans, central Asia), neutral intervals have a long standing acceptance. Within the Greek sphere, 11-L harmony cannot be confined only to the followers of Ptolemy. It seems likely that Aristoxenous chose his temperament partly because of its good simulations of prominent 11-L ratios. Evidence suggests that 11-L harmony was already well established during Hellenistic times. The three-quarter tone and its relatives still enjoy wide usage.

I have examined the relation between 11-L ratios and lesser dimensions in the old diagram: Mapping 11-Limit Ratios and its accompanying page, titled: Why 11-Limit ratios are associated with neutral intervals. I use the same approach that I employed with the 7-L material: map out the most important ratios in a double ‘surrogate’ 5-L matrix, then superimpose it over the actual tri-axial matrix in the two reciprocal positions where the difference is negligible. In the case of the principal 11-L ratios, I paired 11/10 and 12/11 as ‘almost identical’ and as ‘near misses,’ combining the reciprocal matrix patterns into a composite double-strand hexad. This hexad serves as the nucleus for an 11-L
matrix that relates 11-L ratios to the 3-L and 5-L fabric. Not to leave out the 7-L, I made a separate expandable double matrix centered around 11/7 and 14/11. Using this approach I generated some interesting 11-L ratios that sit close to 5-L and 7-L positions.

Looking back at this material I am no longer 100% satisfied. First of all, I no longer consider the difference between 11/10 and 12/11 as 'almost identical.' The ratio 121/120, already seen in the progression of Means, is a double clisma in size. Perhaps under the influence of Aristoxenus, I am now inclined to call 12/11 a three-quarter tone and 11/10 a five-sixths tone. This perspective exposes the crudity of my chosen notation, which I conveniently lifted from extended meantone (31-et) temperament. The notation conflates pitches that are over 14c apart. To be sure, notation poses an ongoing problem for extended just intonation.

Another problem stems from the unfortunate use of the term ‘Harmonic Antipodes’ for the four corners region of the 53-et field. As I explained earlier, the term ‘Harmonic Antipodes’ refers specifically to a property of those equal temperaments that have one circle of fifths. In the case of 53-et, the four corners region is not the Harmonic Antipodes that actually sits in the hook of the flag. That being said, the four corners region is very interesting because in the 53-et system they make the same pitches. They inhabit the same place. These pitches are tending toward neutral intervals but don’t make very good simulations of them. Yet we want comparisons between 11-L norms and the 5-L matrix, not 53-et. Adding to the confusion, the diagram shows the system boundaries for both 53-et and 31-et. In the case of 31-et, the north-east and south-west corners do straddle the Harmonic Antipodes for 31-et. So the term is appropriate with reference to 31-et but not 53-et. No doubt I chose my notation from 31-et because of this positioning, but the 31-et system itself only makes passable (not really good) simulations of some 11-L ratios. Again, our interest here lies with the 5-L matrix and not 31-et.

My willingness to reference 53-et and 31-et, even though the topic is the 5-L matrix, itself shows the workings of Universal Mixture through inter-modelability. I have imposed the morphology of the temperaments over the 5-L matrix because they approximate each other. The question over the appropriateness of this decision hinges on a judgement of tolerance levels. How close is close enough?

I now contend that this diagram can be conveniently simplified. Referring specifically to the three-quarter tone of ratio 12/11 (150.637c or 77s), it sits very close to the 5-L ratio 2048/1875 (152.791c or 78s) in the south-west corner. Its inversion in the octave, the neutral seventh of ratio 11/6, which is the most consonant 11-L ratio, (1049.363c or 535s) sits in the north-east corner with the 5-L ratio 1875/1024 (1047.210c or 534s). The difference again is only 2.154c or about one schisma. The deviations for the metrical value of the other positions can be gauged by schisma and clisma shifts. This approach seems to leave 11/10 (165.004c or 84s) out in the cold, but not really. For it sits close to the 5-L ratio 1125/1024 (162.850c or 83s), in the north-east corner. The difference again is only 2.154c. Its inversion 20/11 (1034.996c or 528s) is equally close to the 5-L ratio 2048/1125 (1037.148c or 529s) in the south-west corner. Other 11-L
noms like 11/8 and 11/9, with their inversions, also have the same minimal deviations. Thus 11-L ratios also display inter-dimensional properties with the 5-L matrix.

One can locate similar inter-dimensional properties between the 11-L and the 7-L. In the diagram, 14/11 (417.508c or 213s) is compared to the 3-L pitch /E of ratio 81/64 (407 820c or 208s). Here the difference of 9.688c or 5s 1 would no longer consider particularly close. It's not a very good example of inter-dimensionality. A better example could be the 7-L ratio 80/63 (413.578c or 211s), reducing the difference to a double schisma. A further improvement can be found in the 5-L ratio 15625/12288 (415.927c or 212s). The point here is that one can always find a closer variant if we inspect the inventory of the lesser dimensions. Given the appropriate level of complexity, a ratio from a higher order can always be well simulated by various contenders from lesser dimensions. This property forms the basis for inter-dimensionality.

This peculiar way in which 'everything is in everything' holds for higher primes as well. For example, the ratio 13/8 (840 528c) sits close to the 5-L ratio 625/384 (843.297c), even closer to the 7-L ratio 729/448 (842.904c). It is superrelatively simulated in the 7-L ratio 512/315 (840.950c). This whole investigation of 'near misses' can be grounded in monochord work, where different divisions produce overlapping results. Awareness of such substitutions forms the likely motivation of progressive tuners to employ higher primes. For example, Eratosthenes used the ratio 19/16 (297.153c) in place of the 3-L version 32/27 (294.135c). Even though the ancients had no apparent name for inter-dimensionality, the concept must have existed in a practical context.

Returning to Anaxagoras, the ancient philosopher, mathematician, and one-time teacher of the great musician Euripides, put much emphasis on Universal Mixture. Anaxagoras refers to the context of inter-dimensionality in his comparison between a 'human' world (5-L) and a highly 'colored' world (n-L). The metaphor acquires its meaning from an old Greek numerical tradition loosely associated with Pythagoreans. The number three is said to be divine, the number five human. The number seven is bestial, or esoteric, or forbidden. The number eleven is never even countenanced. In a related tradition, Pythagoreans were said to value the numbers only up to ten. That would include seven, but it appears that only Archytas among Pythagoreans fostered 7-L harmony. Almost all Pythagoreans argued for strict 3-L norms. Archytas must have been a very unorthodox Pythagorean, if he was one at all. At any rate, the progressive Anaxagoras admitted a potentially infinite inventory of both 'seeds' and 'parts.' In such a milieu the topic of inter-dimensionality comes to the fore.

INTER-COMPLIMENTARITY

My last context for the notion of Universal Mixture takes us back to the vexed relation between reciprocals. Although some harmonies exhibit symmetry, the majority of them prove to be asymmetrical. They form reciprocal pairs or compliments. Such structures can be referred back to the Harmonic Series and Sub-harmonic Series—the ultimate asymmetrical source. They prove to be complete opposites or different worlds, the one mirroring the other. The two sides theoretically have no connection with each
other. Many theorists even deny the very existence of the sub-harmonics. Yet patterns found on one side can also be found on the other side by using the appropriate division. The two divisions thus exchange the upward and downward orientations, making a pair of divisions that I call Duals in the diagrams. Consequently, the harmonics and sub-harmonics interpenetrate each other in a form of ‘everything in everything’ that I term inter-complementarity. Let’s look at some examples.

Take our sample 5-L division 3:4:5:6. As noted, it makes the upward pattern C F A C, downward C G Eb C. Now consider the sequence 10:12:15:20. It exhibits the upward pattern C Eb G C, downward C A F C. The two divisions ruled by 3:6 and 10:20 form a bonded pair that have exchanged their materials between the harmonics and the sub-harmonics. Although the Harmonic Series and its reciprocal have theoretically no connection with each other, yet we see here a commonality—another manifestation of the Heraclitean unity of opposites.

As another simple example, take the 5-L sequence 4:5:6:8. Upward it makes C E G C, on the monochord C Ab F C. Now consider the sequence 15:20:24:30. Upward it goes C F Ab C, downward C G E C. Thus the 5-L sequences ruled by 4:8 and 15:30 form duals of each other, complimentary pairs. Many such examples can be given, illustrations of the doings of inter-complementarity.

It was just such a bonded pair that inspired McClain to do his research. He found that the 5-L Diatonic Heptads ruled by 30:60 and 72:144 have this peculiar relation. He also noted that the inherent materials display a lot of mythological or ‘cosmic’ numbers. McClain proceeded to integrate them into further ‘calendrical’ divisions like 360:720 and yet higher 5-L numbers. In fact, all of the first rate 5-L Diatonic Heptads have significant bonded pairs. 24:48 goes with 90:180. Also 36:72 compliments 120:240. Of course, 3-L harmonics also form duals. For example, the Tetrad 18:36 bonds with 24:48. The Diatonic Heptad 384:768 compliments 486:972. The Diatonic Ogdoad in smallest integers, given by Euclid, 1152:2304 (my D to Db) pairs with 1944:3888. Examples with enormous numbers can also be cited.

However, we must acknowledge that inter-complementarity has a limitation. As we move up the prime number series the gap between the duals grows ever wider. For example, we have just seen that the 5-L sequence 4:5:6:8 has the nearby compliment 15:20:24:30. Let’s transform it into a 7-L harmony with the sequence 4:5:6:7:8. Now the dual can be found in 105:120:140:168:210. As we move into 7-L harmony one can still find the compliment but it involves a more remote division. While the harmonics and the sub-harmonics ‘co-operate’ very well in the 3-L and reasonably well in the 5-L, we see them increasingly distant in the 7-L. A further shift to 11-L harmony makes the gap even more extreme. Yet the principle still holds, the compliment can always be found.

The most instructive way to illustrate this diverging property employs the subsets of the sequence 6:12. The 3-L triad subset 6:8:9:12 is symmetrical, thus what I would call self-dualling. The 5-L tetrad subset 6:8:9:10:12 becomes 30:36:40:45:60. The 7-L pentad subset 6:7:8:9:10:12 has the dual 210:252:280:315:360:420. Finally, the full 11-L hexad
sequence 6:7:8:9:10:11:12 needs even more space. As its dual one finds the more remote division 2310:2520:2772:3080:3465:3960:4620.

We can conclude that inter-complimentarity is always theoretically possible, but not practical with high primes. It would become very messy to find the dual for Ptolemy’s harmony with a 23-L ratio. The Harmonic Series and its reciprocal ultimately diverge. Nevertheless, since most practical (mainstream) harmonies are 3-L and 5-L, the concept is still useful. The form of Universal Mixture that I call inter-complimentarity doesn’t have quite the potency of inter-dimensionality, psychoacoustical equivalence, or inter-modelability. Nevertheless, it shows that the notion of ‘everything in everything’ pops up in the most unexpected places, even here in the examination of asymmetry.

Anaxagoras also discusses the reciprocials and explicitly relates them to Universal Mixture. He says that ‘the large’ (n/1) grows larger approaching the infinite, while ‘the small’ (1/n) grows smaller approaching the infinitesimal. They polarize in the Unlimited. Nevertheless, both sides have ‘equal portions.’ He asserts that they have equal ‘quantity’ or number, and he implies that they have equal status. Crucially, he says that ‘in relation to itself each thing is both large and small.’ This admittedly enigmatic statement supports two forms of Universal Mixture—nesting and inter-complimentarity.

As regards nesting, here is an appropriate example. In 12-et the diatonic semitone is small, one step, the smallest step in the system. But in 31-et the diatomic semitone makes three steps. Now it has become relatively large. In 53-et the diatomic semitone consists of five steps, yet larger. Thus the notion of a diatomic semitone (just norm 16/15) can be both large and small, in relation to the depth of resolution and ‘in relation to itself.’ The statement can also be interpreted to assert inter-complimentarity, since a given ratio ‘in itself’ can be found in both camps, the large and the small. Finally, later in the same passage, he says that here in the third way, which is ruled by division (apokrisis), mixture and separation, everything is still in everything (‘all things are together’) as they were in the first way (‘as they were in the beginning’). He says that Universal Mixture also applies to the third way where it seemingly has no place.

This passage, like so many more segments of the early philosophers, displays great relevance to the subject of Harmonics. Indeed, it makes little sense and becomes downright contradictory when it is removed from the context of Harmonics. It mutates into cryptic problems when dropped into Aristotle’s biological world of ‘material substances.’ Not surprisingly, the era of early Greek philosophy (sixth and fifth century) coincides with the period of the greatest flowering of the ancient Greek musical culture. It was the time of the inspired Orphic poets, the wonderful lyric poets, the crowning dramatists and other famous musicians. In the fourth century musical standards declined and never regained their former heights. But then, in the fourth century, we find the theorists Archytas and Aristothenes.

The status of the reciprocials continues to be an ongoing controversy. Perhaps we shouldn’t be surprised that Anaxagoras’ statements (like those of Heraclitus) prove difficult to interpret. Of course, only a small fraction of Anaxagoras’ book has survived.
He may have dealt with the issue at length somewhere else. At any rate, the available evidence favors support for inter-complementarity.

The same can be said for Archytas, who awarded the very name ‘harmonic mean,’ so implying an equal status for the two Musical Means, AM and HM. Moreover, Aristoxenus favored the highly symmetrical irrational division 72-et, a system that only accentuates inter-complementarity between reciprocals. Consequently, I find it likely that he also recognized the musical principle of inter-complementarity and gave the reciprocals equal weight. Several hundred years later Ptolemy potentially shows a different case, where the use of exotic primes minimizes relevance. Yet Anaxagoras also allowed high primes. Ptolemy uses both upward and downward sequences and his advanced mathematical sense must have made him sensitive to the importance of reciprocals. Alas, no ancient writer has clarified the controversy surrounding the reciprocity between harmonics and sub-harmonics. What status do they have in relation to each other?

THE GREAT MONOCHORD RITUAL

Amongst the diagrams sits an earlier version of the Progression of Means from the early ‘90’s. It covers the same territory as now—the emanation of the successive dimensions. I took it as far as the 11-L. The diagram also illustrates the material on the tri-axial matrix and on the circular graph. However, what has not been included in this diagram also holds interest. The piece has three pages and an extra as a front-piece called ‘the cosmological monochord.’ The assemblage concerns the first six monochord sequences. I confidently called the diagram ‘the first six operations (Emanations) of the sacred Monochord science.’

Pointedly missing from the diagram is any mention of the special relation ratio between the AM and HM, how it generates a progression of significant resolutions—diatonic, chromatic, enharmonic, commatic. At the time I still largely confined the notion of resolution to equal temperaments. I also omitted any reference to tempered analogs. Irrationals were kept off limits. Instead, I put the emphasis squarely on just intonation through the AM, by drawing the monochord for each division. However, on the circular graph I relented by showing the irrational GM, highlighting its symmetry and medial location.

Instead of deriving the Progression of Means directly from the Epimoric Ratio Series, I embedded the epimoric progression within the first division sequences. For example, in the fourth operation, I would now say that the musical fourth 3:4 generates the 7-L level 6:7:8. In the diagram I make the essential mean 6:7:8 part of the overall division 4:5:6:7:8, the whole sequence ruled by 4:8. In short, the six operations show the complete sequences that emerge when the mese sits alternatively at 1, 2, and so on. It commences the field of alternative divisions, the ‘innumerable worlds’ of the early philosophers.
The diagram collects together much, though not all, of the metaphorical associations between numbers, elements as roots, and other references from the literature. For numbers had identifiable associations in the ancient world.

**The monochord** has always had a close partnership with magic. Several interconnected reasons account for it. First of all, music has such a profound effect on consciousness. It can be used to instill an emotion of terror, ecstasy, love, hate, you name it. An instrument that lays bare the building blocks of harmony cannot but become an instrument of power. Secondly, the monochord has long associations with cosmic concerns, planets, and other mysteries. Thirdly, the terminology of magic (opposing forces, roots and compounds) is also the ubiquitous terminology of Harmonics (and medicine, and more!). Even the primary source for magical association, the principal of sympathy or resonance, has a musical base. Fourthly, monochord work engenders active manipulation, a characteristic of magic. It employs the archetypal implement of magic, the moveable bridge (the bronze). The tuner makes the judgement, enacts some division of the One, allowing the gods to be birthed as a living vibratory event. Finally, the monochord has magical value because it emanates aspects of the Sub-harmonic Series, that 'hidden realm' or mirror world. Ancients tended to divide the world into a manifest aspect and an invisible or esoteric aspect approachable by 'signs.' The sub-harmonics rule the 'secret' realm that has perhaps only virtual existence. Such dark occulted issues naturally belong within the purview of magic. Nevertheless, for the ancient commentators magic, science, and medicine cannot be meaningfully separated from each other. All these disciplines use the same metaphorical language of opposing forces and mixtures of elements.

I do not mean to over-emphasize the issue, but only present it because historical writers altogether ignore or suppress the many connections. It forms an undercurrent in the history of musical tuning, strongest during the ancient period. It reminds us that musical tuning has always exhibited esoteric aspects that have a sound numerical base, but sometimes trail off into various occult channels of nonsense. The term 'vibrations' can be abused.

To a large extent the diagram follows the instructions of Plato, who recommended that we examine the musical sequences 1:2, then 2:4, finally 4:8. He doesn't explicitly mention 3, but its interpolation is obvious and implied. I extended the procedure to 5:10 and 6:12, but earlier versions stopped at 4:8. Here we find what I call the cosmological monochord. Nowadays I sometimes call it the elemental monochord, because it makes the first division in which all five classical elements are present. Moreover, they sit in the proper order of density between Heaven (most rare) and Earth (most dense). Additionally, the boundary ratio 7:8 also matches the boundary between consonance and dissonance, giving this division a certain prestige.

In the tuning culture 3-L harmony requires three elements (1,2,3). The 5-L mainstream needs four elements (1,2,3,5). 7-L harmony needs five elements (1,2,3,5,7). We rarely see more elements than five, only among progressives who were probably always a minority. Since the sequence 4:8 has all five classical elements, it provides a
certain closure, especially considering that the next operation 5:10 reveals no new dimension. Plato stops at 4:8 ostensibly because it makes the solid dimension, shown on the diagram in its upward orientation as the tetrahedron matrix of the Harmonic Series Tetr. C E G A#. The harmony itself and its famous 5-L subset C E G and axis C G declares its significance.

The diagram largely speaks for itself, but it also requires some commentary. In the first operation, where the One or the 1:1 becomes 1:2, I neglected to say that this act constitutes the prototype of all further monochord work. Take some measure and put the bridge in the middle, the proto-AM. Take the middle, take the Mean. In monochord work one sees this same action over and over. I should also point out that the mythological context here centers around various stories of the separation of heaven and earth. They form a polarity with a special relation to each other (the octave). The separation creates the space for the matrix of relations, the vortex or mixture, and for a microcosm of the macrocosmic continuum. It defines Anaxagoras' second way, the gateway-controller, the old philosophical term nous. This initial separation between the ancestral elements Fire (Heaven, Stars) and Earth leads to the cosmological monochord 4:8 through the intermediary 2:4. I should also stress that in this first operation the number One, which is androgynous, has given birth to the first female number (even numbers are female).

In the second operation, where 1:2 becomes 2:3:4, the reference to 'the marriage of 2 and 3' is somewhat misleading. It refers properly to the reiteration that makes the growing 3-L Line, but not to the Emanation itself. Instead, the division forms the marriage between the androgynous and the female, giving birth to the first male number. Mythologically it relates to the union of the female with the serpent (representing Time, the highest ruler). Another cosmological context states that the element Air mediates between Heaven and Earth. It separates them, but relates them to each other through the AM. It is the son. It becomes a proxy for the place of any mixture, even the continuum between the octave. The first male number opens the floodgate, making diversity actual. It has many metaphorical associations, centered around the philosophical term psyche. The second operation constitutes the metaphysical justification for 3-L harmony.

In the third operation, 2:3 becomes 4:5:6. The tri-axial matrix emerges. The marriage between the first female and male number results in the 'beautiful male child,' the number 5 as mediator. 5-L harmony needs four elements, but mythological associations also talk of a trinity of three commanding powers. They stem from the primes 2, 3, and 5. Not only are they the masks of the One, they represent the One in action or manifestation. The element Water has wide mythological associations with two main referents. The unmanifest state in the 'Given,' the Silence, is often called the 'primeval waters,' or the waters dammed up or imprisoned. The full harmony is also dominated by the element Water, the waters freed. This is so because, for the mainstream, 5-L harmony is sufficient and complete. Here we see another form of polarity, between the unmanifest and the fully manifest, that comes out poetically as an opposition between Fire and Water, the cosmic summer and winter of the Great Year. By proxy it becomes the opposition between simplicity and complexity, the basis for the opposing forces of Love and Strife. This form of truly dualistic opposition is founded upon the ongoing
progression of the Genera. It is not to be confused with the altogether more subtle unity of opposites between Heaven (Fire) and Earth. Here the opposites unite in cyclical identity.

In the fourth operation, 3:4 becomes 6:7:8. Again, we see the union of a male and a female number to give birth to the male prime 7, the esoteric fifth element Aether. This element has more variations of context than any other, even Water. Sometimes the poets treated it as a stand-in for the element Air. This usage was dominant during the fifth century but lost ground during the fourth century. Alternatively, Aether was also treated as a stand-in for Fire, the place of the stars. This usage appears secondary during the fifth century but rose in popularity during the fourth century. Finally, Aether was also sometimes pictured as an independent element. Archytas perhaps isn't the first. All three usages can be justified on the cosmological monochord 4:8, where Aether sits between Air and Fire. The fifth element remained contentious.

In the fifth operation, 4:5 becomes 8:9:10, the comma resolution. Here I'm no longer satisfied by my commentary. Perhaps because I had nothing to say, I called the sequence 5:10 ‘the Tetractys ratios.’ Now I consider this move an unfortunate diversion from the proper usage of the term. It correctly denotes the familiar triangular pattern that generally connects to the matrix. Sure, it also refers to the Triangular Number Series (1,3,6,10,15,21...) that generates the Tetractys pattern and the interval triangles. This series has musical interest because it tabulates the number of relations between two events (1), three events (3), four events (6) and so on. However musical, it still has little to do with the monochord sequence 5:10 which is, well, just the sequence 5:10.

In the sixth operation, 5:6 becomes 10:11:12. Here again I’m not satisfied with my commentary. Back then I was still largely enveloped in the 3-L milieu. I was in thrall of the Indian modal tunings. Nowadays I would never say that neutral intervals are not used by most tuning cultures. Users include the Balkans, Turkey, Middle East, Iran, Egypt, Sudan, Tunisia, Morocco and more. In addition, I found a ‘blooper.’ The Mousike sequence makes the first 3-L self-dualing harmony, not the first altogether. That distinction, as I knew, goes to the octave. Looking at the diagram now, I’m drawn to the circular graph and its meticulously drawn calculation of some very small intervals. Here we see 11:12 (150.638c or 77s) and 10:11 (165.004c or 84s). They relate by a form of double-cisma 120:121 (14.367c or 7s). Now the mediator GM sits in the middle at 157.821c or 80.5s, only 7.183c or 3.5s from each side. Here we see yet another variant of the cisma. Now I’m rather proud that I managed to draw such a micro interval in that small format.

Perhaps the diagram as a whole was complete with 4:8, the two additions just making an afterthought. At any rate, the prominent position of the fifth element on the cosmological monochord raises the contentious issue of the meaning of Aether among the elements. Two models need to be contrasted with each other. The first and most influential model comes from Aristotle. He conceived the four classical elements as primary material substances or vague ‘great world masses.’ They can be further defined by a permutation using four powers or qualities—hot, cold, dry, wet. For example, Fire is
hot-dry, Water is cold-wet, and so on. He saw the elements as things democratically related to each other. They have equal status. They also have equal capacity to encroach upon each other and gang up on each other. I call this model 'astrological' because it is amenable to a cross of relations like the four seasons or the four compass directions. For Aristotle, the fifth element was a problem—he had to neutralize or tame it. His solution sets Aether apart from the other four elements and takes it to be a pan-element or a sort of inter-element, the space where the four physical elements play. This Aristotelian version of Aether dominates the Hellenistic literature. Modern interpreters anachronistically project it over the early philosophers, a source of confusion. For the early philosophers and poets consistently present a different model of the elements.

I call it 'musical' because it exhibits musical sensibilities. The early fragments always group the elements into two camps: elements of polarity (1,2) and elements of mixture (3,5...). Moreover, the elements don't have an equal status. They exhibit hierarchical relations in a typically musical way. The elements Fire and Earth have priority as well as opposition, reflecting the importance of the octave. Moreover, the old writers didn't artificially restrict the opposing qualities only to hot-cold and dry-wet. The literature also gives light-dark, rare-dense and yet more pairs of opposites. All refer to the potential continuum between the octave, and the special relation of the octave.

Meanwhile, the elements Air and Water are always found together in some mixture between the polar boundaries. The poetic image Air meant mist, fog, cloud—already some mixture of Air and Water. Air as a metaphor is amenable to different 'densities.' Water has the same propensity, sometimes solid, liquid or gaseous. Water and Air tend to merge into each other. This poetic diction suits the interpenetrated relation between 3-L and 5-L harmony.

The element Aether referred originally to the rarified region above the clouds, a special form of Air, closer to Fire or the stars, the region where the planets dwell. Sometimes it's just an aspect of Air or Fire, but sometimes it gains independence. Then it is treated as a valid part of the mixture, consistent with musical expectations. The element Aether specifically denotes 7-L harmony. However, Anaxagoras, who put some special emphasis on the image, gave it a wider meaning. He used it to mean all higher orders of harmony beyond the 5-L mainstream. It refers to the 'highly colored' world of the n-L.

Aristotle took the elements in a literalistic 'scientific' way, but the early writers took it in a poetic manner as a metaphorical language of discourse about just about anything where relation comes into play. Aristotle tried to divide the poets from the philosophers, making the latter 'scientific materialists.' But the philosophers themselves were either outright poets (Xenophanes, Parmenides, Empedocles) or else they wrote in a very colorful prose, in a poetic manner (e.g. Milesians, Heraclitus, Anaxagoras).

Once we re-instate the poetic discourse into the interpretation of early philosophy, its meaning clearly issues forth, its contradictions disappear, its paradoxes find their true foundation, and it becomes self-consistent throughout the movement. Here is an example from Heraclitus. He cryptically says that 'the thunderbolt steers all.' In an orthodox
interpretation, it would mean little more than the primacy of the physical element Fire, and it would apply only to Heraclitus. But from a musical perspective it opens up to reveal much. The lightning strike poetically describes the special relation between Fire and Earth, the unity of opposites. Moreover, the octave 'steers all,' since it acts as a controller of the third way. The statement also refers to Universal Mixture through the word 'steerage.' Finally, it gives the All. This example shows that Heraclitus is the supreme master of compression. He says more of substance with fewer words than any other ancient philosopher. Nevertheless, he is first and foremost of religious figure, a 'doctor of the soul' defending the tenets of orphism.

Some readers may look at this section of my discourse as a diversion away from strict Harmonics toward a speculative corner of ancient philosophy. Not at all. It serves to show how densely interrelated the subjects of harmony and philosophy were in the ancient world. After all, this epoch formed the very peak of the musical achievement. Moreover, philosophy explicitly used musical terms and concepts. The early Greeks drew no distinction between poetry and song. Poetry was generally intoned as 'song-speech' and often accompanied by a musical instrument. Should we be surprised that early philosophy emphasized musical issues and models?

AN UNCERTAIN HISTORY

How much influence did the progressive theorists have on the actual tuning culture of their day? Probably not so much. The large majority of the surviving names of writers support 3-L tuning, including such luminaries as Philolaus and Euclid. This evidence indicates that the 3-L norm was likely to be dominant in actual practice. Yet 3-L and 5-L harmony is so intertwined that the widespread use of 5-L harmony also cannot be denied. The old cosmological numbers confirm it. Nevertheless, the dominant theoretical model hovers around the 3-L line of fifths, extending it out far enough to get an ample inventory of comma-shifted scales. The theoretical language is decidedly 3-L, but the practice also embraces 5-L. The use of 5-L harmony can also be inferred by the occasional historical instances of a matrix model based on the triangle. In a contest between 3-L and 5-L harmony, it's very difficult to say with any certainty which norm is dominant. Of course, the contest itself is essentially meaningless for any line wider than the diatonic, since the difference between the two dimensions amounts to only a schisma shift. For this reason I talk of a 3-L and 5-L mainstream. The move to 7-L ratios feels like a step outside of this mainstream into a more esoteric realm. Consequently, it likely had a small minority position, confirming the boundary of the mainstream.

The progressive theorists crossed boundaries and explored the whole territory in greater depth. They were first of all scientist-mathematicians rather than practicing musicians. How relevant were they after all? We like to think that they gave a good indication of the contemporary tuning culture, of its norms and practices. But in actual fact we must admit that we do not know, and we may never know. Maybe they had considerable influence, or maybe not. The history of musical tuning holds many uncertainties, since very little was written down and much of that was lost. Even the 'safe' notion of a dominant 5-L mainstream has a doubtful aspect. Scholars have

68
established that by Hellenistic times neutral intervals (implying 11-L harmony) were accepted in the Middle East, but the evidence of Aristoxenus hints that they may already have been popular in his own time. We cannot know for sure.

The vagaries of tuning history can be seen in the fortunes of Aristoxenus, that most radically progressive of theorists. Like his teacher Aristotle, he wanted to take music off its pedestal and treat it in a “secular” manner. He wanted to separate it from its old poetical-cosmological-magical-religious associations and put it on an independent scientific base. To this end he wrote in an encyclopedic manner over all aspects of music—its tuning, melodic structure, rhythm, instrumentation, history, and so on. In the late fourth century orphism, magic, and other matters that animated early philosophy had gone quite out of fashion, not to make a notable comeback until a few centuries later. Perhaps Aristoxenus chose his geometrical approach out of this very scientific attitude itself. He wanted to comprehensively map the continuum in a practical manner consistent with atomism—a deep resolution of equal steps.

His brilliance fostered a school of followers that persisted for a very long time. However, it appears that he never had a large following. Moreover, later adherents of his school appear to have had very little real understanding of his work. Perhaps they did not even know that he used geometrical ratios. They just tried to reproduce the specific scales that he had left. They exhibit not appreciation for the great potential of the system to generate and classify new harmonies. They also appear to be totally unaware of the concept of variable resolution, his greatest achievement. Later, Aristoxenus’ parts were taken over by the Byzantine Orthodox Church. Over a period of several hundred years the system further corrupted itself to the extent that a tetrachord has 28 parts (not 30), the octave 68 (not 72). The parts may not be equal at all. Moreover, the diatonic genus was renamed the enharmonic, along with other name changes. Modern musicologists endlessly argue over what tunings were actually used and what (if any) connection they had with Aristoxenus. Again, uncertainty rules.

Ptolemy, who flourished about four hundred and sixty years after Aristoxenus, has a better chance of reflecting the actual tuning practice of his culture, Hellenistic Egypt. The great scientist must have had authority over many subjects. Yet we cannot say whether his tuning philosophy had wide influence or only marginal impact. The most likely story admits considerable authority, though still within a minority position. Since his book is so intensely polemical against Aristoxenus and against the conservative Pythagoreans, it seems to imply a struggle for status. At any rate, his book was faithfully copied for hundreds of years, so it must have had various ups and downs in different places. Ptolemy set the bar very high in the complex world of higher primes. Unfortunately, his followers didn’t realize the potential of his approach to create new harmonies. Instead, they just copied the specific tunings that appear in his book, as if no other alternatives exist. Both Aristoxenus and Ptolemy intended their tetrachords as examples of an integrated approach for further exploration, not as a fixed and restrictive canon. The two progressive schools lost their scientific spirit, shrank into rigidity, and eventually petered out.
In Europe during the renaissance-baroque period (16th to 18th century) Ptolemy regained popularity as an authority, ironically not for his brave embrace of the n-L, rather for his presentation of a particular 5-L diatonic heptad. This harmony was often used as a support for standard meantone and other less well known tunings that largely mimic the 5-L norm. These later Europeans approached Ptolemy from a very different social context. They embraced a form of temperament but used the 5-L norm as an ideal standard to be approximated in the 'real' world by meantone. Ptolemy became the old authority that we love to manipulate into something else.

Ptolemy’s n-L approach with its advanced complexity was bound to be disadvantaged in western Europe, where it eventually fizzled out. However, the school had one more brilliant flowering of creativity in the Arabic Middle East. Al-Farabi in 10th century Baghdad is arguably the most fortunate of all the old western tuning theorists. He sat in the meeting place of three advanced musical cultures: Greek, Arabic, and Persian. Moreover, he had indirect connections with the Indian sphere. He exhibits familiarity with the long line of fifths that justifies the mainstream. He also knew the monochords of Aristoxenus and Ptolemy. As the crowning glory of his extraordinary eclecticism, he gave some tunings both in arithmetical ratios and in Aristoxenean parts. This evidence implies that he could translate between them—that he could (like Anaxagoras) transcend the 'sectarian' divide between the rational and irrational camps. Moreover, he shows the ability in his collection of tunings that he can use the system of parts as a scaffold in which to organize arithmetical options. (See my paper: Homage to Ptolemy, where I bring together Aristoxenus, Ptolemy, and al-Farabi). The use of tempered parts is analogous to the modern use of tempered cens and my use of tempered schismas. It provides a language in which to atomize the continuum so that the metrical properties of arithmetical ratios can be gauged.

Al-Farabi demonstrated the potential of this integrated approach with some new n-L harmonies that follow a particular sequence in the system of parts. His approach was further clarified, confirmed, and extended by his brilliant 11th century follower, Ibn-Sina, who particularly favored 13-L harmonies. One more name in the 12th century, the theorist Saffiyu-d-Din, added some more original contributions before the stream dried up. Thus the school of Ptolemy had its last creative growth in the medieval Middle East. Perhaps we should not be surprised, since Ptolemy himself lived in Egypt and strove to integrate Greek and Babylonian science.

The work of al-Farabi and his followers is even more likely to reflect his actual contemporary practice than the older work of Ptolemy. Yet many uncertainties remain. The history of tuning in the Middle East is fraught with ambiguities and controversies. Sometimes the old 3-L line of fifths hold sway among writers, but 3-L and 5-L norms are not necessarily an impediment to a more variable practice. The same situation holds for the modern Turkish sphere. Most Turkish theorists extol a comma resolution approaching 53-et, but actual practice is more variable and regional. Again, uncertainty is the order of the day.
This uncertainty is not restricted only to the western world. Take the case of ancient China. This society took musical issues quite seriously and at a state level. They instituted a standard pitch—something the Europeans didn’t do until the 19th century. Also, the ancient Chinese produced the oldest published descriptions of monochords. Their theoretical language holds the 3-L, and they knew the Dodecachord. Yet a sculpted bronze kinn (monochord) has been found that marked the ratios 5/3, 5/4, 6/5 and 8/7. Consequently, it has become difficult to confirm that the Chinese culture always tuned strictly 3-L. The pythagorean norm likely dominated, but doubts about actual practice still surface, especially since the Chinese world was not homogeneous. We recognize various regional musical cultures. The evidence, though admittedly scant, suggests pluralism.

In India, the oldest and most authoritative text on tuning was not written until Hellenistic times, but it probably reflects practice that was already very ancient. The text defends the mainstream, using the comma (sruti) as the basic building block. The Natyashastra does not clearly articulate a method. Instead, it throws up puzzles and problems. It appears to be intentionally written in such a way as to confuse the uninitiated. Only someone who already understands musical tuning can make sense of it. The text never mentions a monochord, or even ratios. Nevertheless, it proposes a cryptic experiment comparing the fret positions for two vinas (fretted lutes, that is, applied monochords!). The experiment is supposed to prove that there are exactly 22 srutis in the octave. But the octave actually contains 53 commas, not 22. Rather, there are 22 commas in the tetrachord, which formed the basic scalar module like in the west. This puzzle can be solved if we reinterpret the twenty-two as a long line of fifths. Ignoring schisma shifts one generates what I call the ‘triple strand’ subset of the wider matrix. The model fits the musical practice where semitones of three, four, or five commas prevail, but enharmonics (of two commas) are restricted to use in ornamentation rather than scale degrees. It also suits the confusing statement that the sruti comes in different sizes (one, three, and four commas). This lack of clarity in the text is typical of many (most?) writings on musical tuning. Both in the east and the west, the subject of musical tuning tends to attract obscurity, puzzles, paradoxes, and other difficult issues.

The late Romans gave the restrictive 3-L Diatonic Octochoord to the western Catholic Church, where it served as a norm unchallenged until the 14th century. Here we have what appears to be the longest period of tuning monism in history, almost a thousand years. Yet this picture is also mostly illusory. The medieval tuning theorists hailed as churchmen, and only church-related music was written down at all. The tuning paradigm served more as an ideology, apparently a means to eliminate any trace of 5-L ratios. However, the actual practice is far less certain. Music also existed that was not sanctioned by the Church and not written down, for example dance music and work music. Moreover, in many isolated and/or backward parts of Europe, pockets of cultural diversity transgressed the sanction of the 3-L line of fifths, even into modern times. For example, consider the old choral tradition found on the island of Corsica. In addition, specific traditional musical instruments continued to have associated tunings that didn’t necessarily please the pythagoreans. Take, as an example, the Scottish bagpipes, whose scale varies markedly from the paradigm. In short, the theory of a long-lived 3-L monism
is a tidy simplification. The reality of actual tuning practice is at once more complex and less certain. In Europe we see a patch-work of many local cultural traditions at variance with one another.

The 14th century expansion of the 3-L line let the 5-L norms into the house through the back door. By the 15th century most musicians preferred the 5/4 major third over the comma-shifted dissonance. The stage was set for a uniquely extraordinary transition from arithmetical ratios to a temperament—standard meantone. The first explicit theoretical descriptions appear early in the 16th century, but it was by then already in practical use for some time. We do not know exactly who discovered it or where or when. Yet another uncertainty in the history of musical tuning, the origins of meantone have intrigued me for a long time. I looked into it, planning to write a paper on the subject—but it never happened. I have a hunch but little hard proof. My theory speculates that it once inhabited a local music culture of southern Britain and/or Wales and then was exported to the continent during the early 15th century with the composer John Dunstable and others. Some circumstantial evidence supports this theory. According to various writers the harpsichord had always been tuned meantone. One can trace the origins of the harpsichord to 14th century Britain. Also, Wales had a vocal tradition of singing in parallel thirds. Moreover, English theorists, even in the 13th century, gave enhanced status to the 5/4 as a consonance. Meantone is essentially an attitude of support for the 5/4 and the willingness to compromise the 3/2 a little to achieve it.

I also tried to find the origins of meantone through another method. By looking at a musical score one can judge if it would work in meantone rather than 3-L. Using the standard meantone line of fifths from Eb to G#, certain chords are dissonant and to be avoided. If the score avoids these very chords, then it is a potential candidate for the meantone orientation. Additionally, certain chord progressions work in meantone but when actualized in the 5-L resolution they generate unwanted comma juxtapositions. When such progressions dominate a score, it gives a sign that the meantone milieu is in the air. Using these criteria I tentatively put the transition somewhere around the middle of the 15th century. However, I admit that this game is fraught with uncertainty. It involves making aesthetic judgements that may not have applied to the actors themselves. In my subjective judgement, I hear early compositions of Dufay as latently medieval, but his late work as clearly renaissance. Another musicologist may disagree. Nevertheless, we all agree that the 15th century (whether early or late) marks a watershed.

Standard meantone tuning dominated the 16th and 17th century, especially on keyboards, but it was never the only game in town. Various tuning theorists defended the 3-L norm or else gave alternative forms of meantone. However, these variant forms of tuning never had much influence, since they were generally more difficult to tune and had other shortcomings. Standard meantone proved to be the best available option. Some instruments, e.g. small fretted lutes like mandolins, were out of necessity set in something like 12-et or 3-L. Large lutes proved to be extremely popular among both professionals and amateurs. Here we see an interesting picture of uncertainty that shows a microcosm of the history of musical tuning in general.
Lutes have tied-on and moveable frets. Consequently, each player can customize the scale as desired. Most players did not write out their tuning in any way. Likely they knew very little about the mathematical theory and did it intuitively 'by ear.' However, a small number of players did communicate their system, through a monochord, ratio table or other means. We also have some evidence from paintings, etc. It appears that some players set their frets to the meantone norm, occasionally even adding an enharmonic fret or two. Some conform to a 3-L norm. But the majority (e.g. Dowland) used a personalized compromise between meantone and 3-L. Thus we see a milieu of considerable complexity and uncertainty. Many varieties coexist that can only be pinned down with difficulty. The late 16th century theorist Vincenzo Galilei (father of the scientist) recommended a string of 18/17 ratios. The fact that it doesn't add up to an octave made no difference to him, since the lute is fretted only to the ninth fret. In short, the lute became the focal point of much experimentation.

Meantone tuning exhibits two tendencies, restrictive and expansive. We have noted the restrictive aspect. If we want a major chord on B it becomes dissonant because we must use an Eb rather than the desired D#. The meantone line of fifths only artificially stops at twelve members. By the early 16th century people began building keyboards with 'split keys' so that one could have both Eb and D#, both Ab and G#. The expansive tendency took hold. Around the mid 16th century the avant-garde composer-theorist Nicola Vicentino made a keyboard for all thirty-one notes, but it never took hold because it was too difficult to play. Here we see the achilles' heel of meantone tuning. No one could think of a practical keyboard for a full enharmonic resolution. A workable design only appeared around 1870 with R.H.M. Bosanquet, but he ironically intended it for 53-et rather than 31-et. Meantone tuning appears first and foremost a keyboard tuning, and keyboards became increasingly popular through the 17th, 18th, and 19th century. Players increasingly turned to an alternative, well temperament, as a tuning norm that better suits the old twelve-note keyboard design.

The roots of well temperament may also lie in the 15th century. The first writer to give tuning directions that can be interpreted as appropriate came from the German Arnold Schlick early in the 16th century. Well temperament, and temperament in general, was given a tremendous boost by the discovery of logarithms in the 17th century. During the 18th century one finds many tuning theorists with a whole variety of well temperaments, irregular temperaments, and multiple divisions. (See my paper: Irregular Temperaments and the Division of the Ditonic Comma). It appears that well temperaments were first regionally popular in northern Germany, where one-sixth syntonic comma meantone also enjoyed some popularity. This specific alternative meantone tuning is particularly amenable to tweaking into a good well temperament. By the 18th century well temperament dominated over meantone in some parts of Europe. In the 19th century meantone continued to lose ground, although it never died out completely. Meanwhile, well temperaments became increasingly shallow so that over time they sounded indistinguishable from equal temperament.

In the 19th century aftermath of the industrial revolution, theorists strove for the standardization of intonation for all the orchestral instruments, using the piano as a
reference, in this manner one can have a large orchestra that is pan-European. Equal temperament became the industrial ideal, although its rigorous application to keyboards didn’t take hold until certain techniques by professional tuners became standard around the first World War. Europe and its colonies entered into another period of tuning monism.

However, this monism began to crumble by the second half of the 20th century. Better communications resulted in more collaboration between disparate tuning cultures. In my case, the tunings of India opened the door to tuning-in-general for me. In addition, increased interest in historical accuracy leads many musicians to restore older practice. One may want to perform Handel’s keyboard music in his own tuning (meantone) and pitch. Thirdly, the proliferation of new electronic keyboards that are essentially computers has made tuning modifications easy. Tuning and timbre have become amenable to an open-ended exploration. The future appears bright for pluralism.

This brief summary of the history shows that considerable uncertainty divides theoretical models and the actual practice. Such an environment only encourages further polemical disputes between opposing parties of theorists. The polemics have permeated the whole history. Euclid and Ptolemy disparaged Aristoxenus. The 16th century Zarlino favored the 5-L ideal over 3-L but accepted the practical meantone. In the 17th century Marin Mersenne controversially accepted 7/6 as a consonance. Nicola Mercator favored 53 commas. Andreas Werckmeister fostered well temperament. Christiaan Huygens supported 31-et. In the 18th century Rameau defended Zarlino but stressed the hegemony of the Harmonic Series over its reciprocal. In the 19th century Hermann Helmholz argued for 53-et. And so on and on. Such disputes continue today, usually oriented around the main fault-line between rational and irrational ratios. In my opinion such polemics constitute a waste of energy, due to the workings of Universal Mixture.

COMPROMISE AND JUDGEMENT

At its core, musical tuning comes down to a judgement over tolerance levels. How close is close enough? Our decisions on these matters orient us toward a meantone, an irregular temperament or whatever. Different tunings re-assign the compromises between ratio norms in different places. If the package of compromises suits our judgements concerning tolerance, then the tuning system is deemed acceptable.

Tuning systems form variants of each other by shifting priorities. In the 3-L diatonic (seven note) scale, the fifths are rewarded but the thirds pay a heavy price. In the 5-L diatonic scale we favor both the good fifths and thirds, but one fifth (between D and A) must become glaringly dissonant. It forms the sacrifice (compromise) that permits the others to remain just. No matter what our strategy, we cannot avoid making some series of compromises. When we use the meantone diatonic scale one has gotten rid of the dissonant fifth of the 5-L, but in order to do it all of the fifths need to be compromised a little (a quarter of a syntonic comma flat). Here the fifths have paid a price so that the thirds can be just. In making this adjustment we gain as a bonus some good 7-L simulations appropriate to an enharmonic resolution. The fifths have only paid a small
price, usually judged to be worthwhile. A heavier burden in the temperament is dumped on the whole tone, a half-comma between 10/9 and 9/8. This fortunate placement of the compromise has always been an argument in favor of meantone—since the whole tone is a (soft) dissonance anyway, we can better place the bigger compromise here and save the thirds.

Every tuning system can be defended (or attacked) by some argument about the judicious placement of compromises.

Compromise pervades all aspects of musical tuning. It shows up between the levels of resolution in multiple divisions. In 12-et one sees a shallow resolution that offers the gift of great simplicity, but at the cost of considerable compromise in the purity of the ratio norms. By contrast, in 53-et a deep resolution vastly improves the purity of the 5-L ratio norms, but it comes at the cost of complexity—many tones to the octave. A workable keyboard for 53-et is a much harder design problem (now solved) than a keyboard for 12-et. Here the trade-off lies between simplicity and accuracy.

Compromise also pervades the very dimensions of arithmetical ratios. In 3-L harmony a rapid increase in numerosity soon leads to enormous ratio numbers. Now 5-L harmony tames the numbers considerably. 7-L even more. This property encourages the progressive tuner to use a high prime sequence such as 16:17:18 in order to keep down the overall monochord numbers. But here too a trade-off comes into play. Every new dimension introduces new forms of dissonance and enhanced complexity of interval type. Moreover, each new dimension deconstructs symmetry into asymmetry. If we value maximum symmetry we should stay with the 3-L. Of course, if we value asymmetry then high primes become attractive. The point here is that a tweaking of one parameter in a tuning system always results in some adjustments in other parameters. Compromise rules.

The question of judgement and the making of compromises between competing norms already appeared very early in Greek philosophy. Speaking of the ‘innumerable worlds’ that morph into each other, Anaximander says that ‘they pay penalty and retribution to each other for their injustice in accordance with the ordering of Time.’ It happens ‘according to necessity’ because the properties of number relations are exact and demonstrable. Anaximander spearheads the radically progressive strain of early Greek philosophy. Here the elements (roots) that sustain the ‘things that exist’ are not confined only to the traditional poetic images of fire, earth, air and so on. Rather, they also include ‘the unlimited’ (apeiron), the All, the continuum, the union of rationals and irrationals. He gropes for an analog conception of harmony in which the One is tantamount to the All. It becomes the ‘first way’ of the Eleatics and Anaxagoras. For Anaximander the ‘innumerable worlds’ can be distilled or extracted (‘separated out’) from the All by using the appropriate division, thereby setting the compromises in the relevant places.

Matters of judgement forms a prominent theme throughout early Greek philosophy. This theme suits a milieu of intonational pluralism.
Later in the 6th century Lassus also addressed the issue in his book on music. He said that the fret positions on a monochord have 'breadth.' The ratio position is more like a smear or a small line segment than a mere point. In other words, ratios are amenable to some leeway, some play in their positions. They have a zone of tolerance that allows compromises to be made. The width of this zone of tolerance courts controversy, contributing to conflicting schools of tuning culture.

I will end this essay with a quote from Heraclitus, who is arguably the most musical of the ancient philosophers. With his usual compression he addresses a number of themes that impinge on the issue of compromise and judgement. He says that 'things taken together are both whole and not whole, being brought together and brought apart, in tune and out of tune, out of all things there comes a unity, and out of a unity comes all things.' The 'things taken together' refers to the All or the field of all possibilities, Anaximander's 'innumerable worlds,' They are 'both whole and not whole' because in musical materials parts are also wholes. They can be 'brought together and brought apart, in tune and out of tune' according to where our judgement places the compromises. Our decisions generate the competing systems of harmony. These possibilities define the countless ways in which the One and the Many relate to each other as vibratory events.

SUMMARY

In this essay I have tried to explain why the subject of musical tuning has given rise to controversy, polemics, and competing schools of approach. In order to get an overall perspective I needed to bring out some of the structural or morphological differences between various tuning systems. But the arithmetic itself is secondary to an appreciation of the underlying principles or foundations. Here is where the 'problems' originate. Fortunately, the number of pillars proves to be surprisingly small. I present them here, in no particular order. Prioritizing them proves to be difficult. They are all important and have numerous ramifications between themselves and further issues.

1. Musical ratios come in two sorts: rational (arithmetical) and irrational (geometrical).
2. The Harmonic Series is structurally asymmetrical, implying the Sub-harmonic Series. Reciprocity rules.
3. The Prime Number Series commands the field of arithmetical ratios.
4. Harmonies (relations) naturally model themselves as sets that have within them subsets and supersets. Matrices abound.
5. Harmonies form wholes within wholes.

The first pillar has led to the ongoing battle between the proponents of temperament and the followers of just intonation. But tunings can also be made that mix them together. Moreover, one camp can simulate the other camp—the basis for the musical principle of psychoacoustical equivalence. The battle goes on but it's pointless.

The second pillar leads to the unresolved issue of the status of reciprocals. Some contend that sub-harmonics have no reality at all. Perhaps they have only a virtual existence. Yet we plot them on a monochord and play them. Reciprocity pervades other
aspects of the field, such as pitch direction. Why deny it to the Harmonic Series?
Evidently we can derive the arithmetical ratios from the Harmonic Series and/or the Sub-
harmonic Series, through the agency of the two reciprocal Musical Means (HM and AM).
Geometrical ratios come through the independent agency of the GM. Thus musical ratios
need not come solely from the Harmonic Series alone. Harmonics seem to present the
perceptible realm, while sub-harmonics present a hidden or mirror world. Symmetrical
harmonies straddle both sides. Although some would consider the hidden side
superfluous, it must have some value, since the two sides can always exchange patterns
between each other—the basis for the musical principle of inter-complimentarity. The
notion of reciprocity also proves to be very useful in constructing a consistent music
theory. Consequently, I would follow the classical tuning theorists and treat the
reciprocals as if they are equal, even though they do not appear to be so in the perceptible
realm. This game of pretend surrounds the central enigma of Harmonics—the status of
reciprocals. Here the controversy is real.

The third pillar has given rise to competing schools of just intonation tuning:
those that favor 3-L (pythagoreans), 5-L (the mainstream), 7-L (progressives) and so on.
Yet one also finds that the various dimensions can simulate each other—the basis for the
musical principle of inter-dimensionality. Thus this competition also has no point. Prime
numbers as originate elements ‘marry’ to generate composite numbers. They join
together to make ratio sequences of whatever complexity. Numbers combine through
least common multiples to construct higher edifices of relationship (logos). Fundamental
to this architecture is the Epimoric Ratio Series that generates the Progression of Means.
Using these tools we can map the field of arithmetical ratios. Geometrical ratios seem to
exist entirely independent of this basis, but in reality they occupy the same
psychoacoustical space. They interpenetrate the territory of each other, so that isolation
from each other is ultimately futile.

The fourth pillar has been less contentious. Since harmonics define patterns of
relationship, these patterns can best be modelled as a matrix. An arithmetical harmony
always consists of some mixture of prime roots and composites made to co-habit some
sequence such as the octave or tetrachord. Within the sequences one can always find
various subsets, and the sequence itself can also form an aspect of yet more inclusive
alternative supersets. Geometrical harmonies substitute irrational ‘parts’ or ‘steps’ for the
scale in place of the traditional ratios, but these geometrical positions can be treated in an
analogous way. Steps are amenable to aggregation into compound structures, an
alternative to ratio sequences as a means to measure harmonic space. In both cases we
talk about subsets and supersets.

The fifth pillar has been largely ignored and perhaps not well understood. Since
subsets and supersets form parts which are also architectonic wholes within wholes,
various peculiar properties result that can be gathered under the general heading
‘Universal Mixture.’ These properties have not been widely appreciated, but they were
certainly pondered by the early Greek musical philosophers. Unfortunately, they used
them mostly to express a paradox or a mystical stand that relates mostly to the ‘first way’
and the prestigious gate of the ‘second way.’ The problematic statement by Anaxagoras
that it also applies to the 'third way' prompted me to investigate. The results only enhance my appreciation of the early philosophers, for their sensitivity to the implications of a musical model of the world.

AFTERWORDS

In this essay I have endeavored to explain a rather complex subject in such a way that it can be understood by the lay person or non-specialist. At the same time I've tried not to avoid the more difficult issues and not to 'dumb down.' I leave the reader to decide whether or not I've been successful. Hopefully, my efforts will equip the explorer with valuable tools to approach my charts and diagrams with more intelligence. The work can also serve as a companion to some of my other essays that delve more deeply into specifics. Lastly, I hope that my essay helps to de-fuse the poisonous atmosphere that still prevails whenever the subject of musical tuning is raised. Too much heat and not enough light. Cooperation rather than competition promises a further joyous exploration of the 'innumerable worlds.' Let tolerant pluralism reign!

- November, 2015, Amsterdam