Analysis of a series RLC circuit using Laplace Transforms Part 2.

First a quick reminder of the circuit (Figure 1):

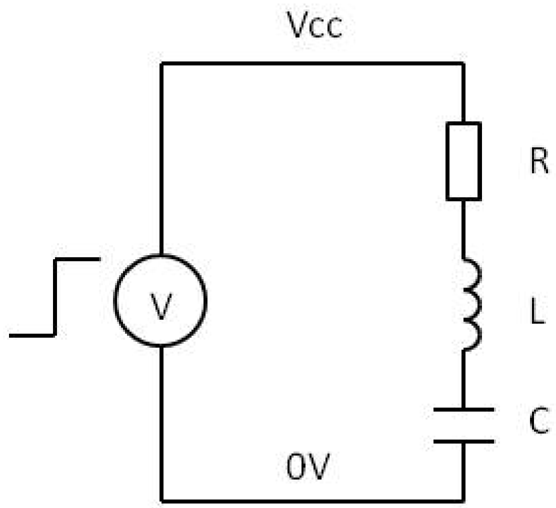

Figure 1 The series RLC circuit driven by a step change in voltage

I'll demonstrate how to derive the voltage across each component, in the time-domain, starting with the easy one first…

Voltage across the resistor, $V_R(t)$.

You will recall from Part 1 that we derived the following expression for i(t)

$$i(t) = \frac{V}{L}.\frac{1}{\sqrt{\omega_n^2 - a^2}}.\sin(\sqrt{\omega_n^2 - a^2})t$$

This was equation (11) in Part 1.

For simplicity, I will rewrite this as

$$i(t) = \frac{V}{L}.\frac{1}{b}.\sin(bt) \qquad (1)$$

where

$$b = \sqrt{\omega_n^2 - a^2}$$

$$a = \frac{R}{2L} \qquad (2)$$

and

$$\omega_n = \frac{1}{\sqrt{LC}} \qquad (3)$$

Because R is a simple, non-reactive resistance the expression for V_R is therefore simply

$$V_R(t) = \frac{VR}{L}.\frac{1}{b}.\sin(bt) \qquad (4)$$

That's it!

Voltage across the capacitor $V_C(t)$ – Laplace method.

I'll tackle $V_C(t)$ next as in a real-life circuit this is the voltage which you are most likely to want to measure, probably with an oscilloscope. This is because $V_C(t)$ is the voltage across the closed switch in the half-bridge circuit, as shown in Figure 2.

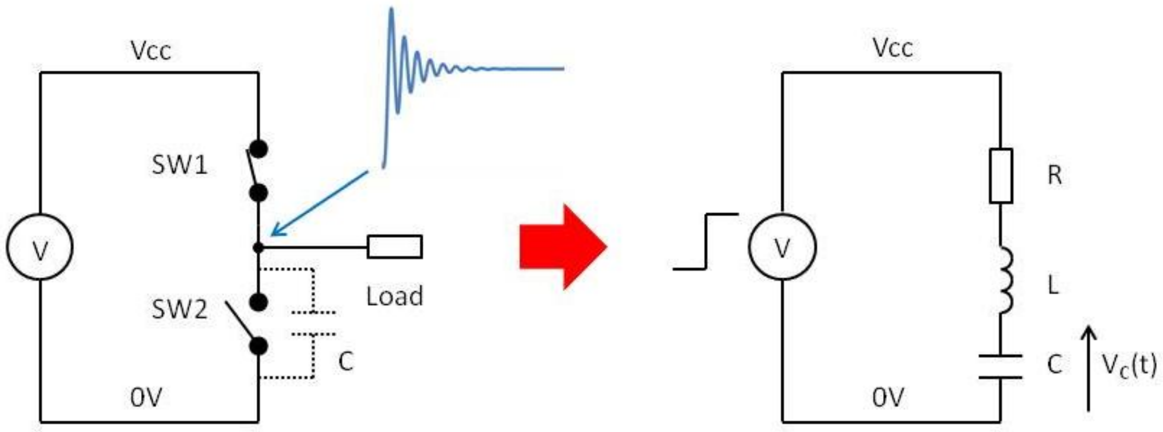

Figure 2 $V_C(t)$ - the voltage across closed switch SW2

I'll show two methods of deriving an expression for $V_C(t)$: an easy method (Laplace) and a less-easy method (calculus). Hopefully you'll agree with me as to which is which!

So, Laplace method first. In part 1 we derived an intermediate expression for i(s) which was:

$$i(s) = \frac{V}{s\left(R + sL + \frac{1}{sC}\right)} \qquad (5)$$

In the s-domain we can also say that

$$V_C(s) = i(s).Z_C(s) \qquad (6)$$

where $Z_C(s)$ is the "Laplace impedance" of C i.e.

$$Z_C(s) = \frac{1}{sC} \qquad (7)$$

Substituting (5) and (7) into (6) we have

$$V_C(s) = \frac{V}{s\left(R + sL + \frac{1}{sC}\right)sC} \qquad (8)$$

and now we're at step 4 of the process described in Part 1 – rearranging (8) to fit one of the standard transform pairs found in the table at:

http://www.me.unm.edu/~starr/teaching/me380/Laplace.pdf

With a little manipulation we can rewrite (8) as:

$$V_C(s) = \frac{V}{LC} \cdot \frac{1}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} \qquad (9)$$

where V, L and C are constants and can be taken out of the transformation process.

Is there an appropriate transform pair that will allow us to transform

$$\frac{1}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

back into the t-domain? The answer is yes – number 27, which is:

$$L^{-1}\left\{\frac{1}{s\left[(s+a)^2 + b^2\right]}\right\} = \frac{1}{a^2 + b^2} + \frac{1}{b\sqrt{a^2 + b^2}}.e^{-at}\sin(bt - \phi) \qquad (10)$$

where

ϕ = atan2(b, -a)

This looks a little daunting, but there’s actually quite a lot of simplification which can be carried out. So…

$$V_C(t) = \frac{V}{LC}\cdot\left[\frac{1}{a^2 + \omega_n^2 - a^2} + \frac{1}{\sqrt{\omega_n^2 - a^2}.\sqrt{a^2 + \omega_n^2 - a^2}}.e^{-at}\sin(bt - \phi)\right]$$

$$V_C(t) = \frac{V}{LC}\cdot\left[\frac{1}{\omega_n^2} + \frac{1}{b\omega_n}.e^{-at}\sin(bt - \phi)\right]$$

$$V_C(t) = \frac{V}{LC}\cdot\left[LC + \frac{\sqrt{LC}}{b}.e^{-at}\sin(bt - \phi)\right]$$

$$V_C(t) = V\left[1 + \frac{\omega_n}{b}.e^{-at}\sin(bt - \phi)\right] \qquad (11)$$

Dimensionally this is consistent, as everything in the square brackets is dimension*less*, and hence the units on both sides of the equation are volts. A comparison of results obtained from (11) with results from an LTSPICE simulation (R = 1Ω, L = 5nH, C = 200pF) confirm identical results from the two methods.

Voltage across the capacitor $V_C(t)$ – calculus method.

Given that we already have an expression for i(t) – equation (1) above – we can also derive an expression for $V_C(t)$ using calculus rather than further use of Laplace Transforms.

For any capacitor we can say that Q = C.V_C

where

Q is the charge on the capacitor

V_C is the voltage across the capacitor

C is the capacitance of the capacitor

We can also say that $Q = \int_0^t i\,dt$

So

$$V_C(t) = \frac{Q}{C} = \frac{1}{C}.\int_0^t i\,dt \qquad (12)$$

Substituting (1) into (12) we have

$$V_C(t) = \frac{V}{L}.\frac{1}{b}.\frac{1}{C}\int_0^t e^{-at}\sin(bt) \qquad (13)$$

V, b, L and C are constants, and therefore we simply need to integrate the expression $e^{-at}\sin(bt)$ between the limits 0 and t and we will have an expression for $V_C(t)$.

$e^{-at}\sin(bt)$ can be integrated by parts and is of the form:

$$\int v\frac{du}{dt} = uv - \int U\frac{dv}{dt}$$

Let $\quad v = e^{-at} \therefore \frac{dv}{dt} = -ae^{-at}$

$$\frac{du}{dt} = \sin(bt) \therefore u = -\frac{1}{b}\cos(bt)$$

$$\therefore uv - \int u\frac{dv}{dt} = -\frac{1}{b}\cos(bt).e^{-at} - \int -\frac{1}{b}\cos(bt).-ae^{-at}$$

$$\boxed{\int e^{-at}\sin(bt) = -\frac{e^{-at}}{b}\cos(bt) - \frac{a}{b}\int e^{-at}\cos(bt)} \qquad (14)$$

Now we need to integrate again – this time the term $e^{-at}\cos(bt)$

Let $\quad v_1 = e^{-at} \therefore \frac{dv_1}{dt} = -ae^{-at}$

$$\frac{du_1}{dt} = \cos(bt) \therefore u_1 = \frac{1}{b}\sin(bt)$$

$$\therefore u_1 v_1 - \int u_1 \frac{dv_1}{dt} = \frac{1}{b}\sin(bt).e^{-at} - \int \frac{1}{b}\sin(bt).-ae^{-at}$$

$$\boxed{\int e^{-at}\cos(bt) = \frac{e^{-at}}{b}\sin(bt) + \frac{a}{b}\int e^{-at}\sin(bt)} \qquad (15)$$

Substituting (15) into (14) we have:

$$\int e^{-at}\sin(bt) = -\frac{e^{-at}}{b}\cos(bt) - \frac{a}{b}\left[\frac{e^{-at}}{b}\sin(bt) + \frac{a}{b}\int e^{-at}\sin(bt)\right]$$

$$\int e^{-at}\sin(bt) = -\frac{e^{-at}}{b}\cos(bt) - \frac{ae^{-at}}{b^2}\sin(bt) - \frac{a^2}{b^2}\int e^{-at}\sin(bt)$$

$$\left[1 + \frac{a^2}{b^2}\right]\int e^{-at}\sin(bt) = -\frac{e^{-at}}{b}\cos(bt) - \frac{ae^{-at}}{b^2}\sin(bt)$$

$$\int e^{-at}\sin(bt) = \left[1 + \frac{a^2}{b^2}\right]^{-1} . \left[-\frac{e^{-at}}{b}\cos(bt) - \frac{ae^{-at}}{b^2}\sin(bt)\right]$$

$$\int e^{-at}\sin(bt) = \frac{b^2}{a^2 + b^2}\left[-\frac{e^{-at}}{b}\cos(bt) - \frac{ae^{-at}}{b^2}\sin(bt)\right]$$

$$\int e^{-at}\sin(bt) = -\frac{e^{-at}b^2}{a^2 + b^2}\left[\frac{1}{b}\cos(bt) + \frac{a}{b^2}\sin(bt)\right]$$

$$\int e^{-at}\sin(bt) = -\frac{e^{-at}b^2}{\omega_n^2}\left[\frac{1}{b}\cos(bt) + \frac{a}{b^2}\sin(bt)\right]$$

$$\int e^{-at}\sin(bt) = -\frac{e^{-at}}{\omega_n^2}\left[b\cos(bt) + a\sin(bt)\right] \qquad (16)$$

Substituting (16) into (13):

$$V_C = -\frac{1}{C}.\frac{V}{L}.\frac{1}{b}.\frac{e^{-at}}{\omega_n^2}\left[b\cos(bt) + a\sin(bt)\right]_0^t$$

but $\omega_n^2 = \frac{1}{LC}$, so

$$V_C = -\frac{Ve^{-at}}{b}\left[b\cos(bt) + a\sin(bt)\right]_0^t$$

Applying upper and lower limits:

$$V_C = -\frac{Ve^{-at}}{b}\left[b\cos(bt) + a\sin(bt)\right]_0^t + \frac{V}{b}[b]$$

$$V_C = V\left[1 - \frac{e^{-at}}{b}\left[b\cos(bt) + a\sin(bt)\right]\right] \qquad (17)$$

also:

$$a\sin(bt) + b\cos(bt) = -c\sin(bt - \phi)$$

where

$$c = \sqrt{\left(a^2 + b^2\right)}$$

$$\phi = \tan^{-1}\left(-\frac{b}{a}\right)$$

so

$$a\sin(bt) + b\cos(bt) = -\sqrt{a^2 + b^2}\,\sin(bt - \phi) \qquad (18)$$

Substituting (18) into (17):

$$V_C = V\left[1 + \frac{\sqrt{a^2 + b^2}}{b} e^{-at}\sin(bt - \phi)\right]$$

but $b^2 = \omega_n^2 - a^2 \therefore \sqrt{a^2 + b^2} = \sqrt{a^2 + \omega_n^2 - a^2} = \sqrt{\omega_n^2} = \omega_n$

$$\therefore V_C = V\left[1 + \frac{\omega_n}{b} e^{-at}\sin(bt - \phi)\right] \qquad (19)$$

which is exactly the result we obtained using the Laplace method – equation (11). Note that the expression for ϕ is a little different for the Laplace method: ϕ = atan2(b, -a) compared to $\phi = \tan^{-1}$(b, -a) for the calculus method. The difference between atan2 and $\tan^{-1}$ (or atan) is very well explained in this article.

I have deliberately shown all my working, for both methods, in order to demonstrate the relative simplicity of the Laplace method compared to the calculus method. At the very least, there is much more scope for typographical errors using the latter.

Voltage across the inductor $V_L(t)$ – Laplace method.

I'm now going to derive expressions for $V_L(t)$ in the same way as for $V_C(t)$ using first the Laplace method, then the calculus method, to again illustrate the simplicity of the former compared to the latter. Starting with the Laplace method …

In part 1 we derived an intermediate expression for i(s) which was:

$$i(s) = \frac{V}{s\left(R + sL + \frac{1}{sC}\right)} \qquad (20)$$

In the s-domain we can also say that

$$V_L(s) = i(s).Z_L(s) \qquad (21)$$

where $Z_L(s)$ is the "Laplace impedance" of L i.e.

$$Z_L(s) = sL \qquad (22)$$

Substituting (20) and (22) into (21) we have

$$V_L(s) = V.\frac{sL}{s\left(R + sL + \frac{1}{sC}\right)}$$

after some re-arrangement…

$$V_L(s) = VL.\frac{s}{s\left(R + sL + \frac{1}{sC}\right)}$$

$$V_L(s) = V.\frac{s}{\left(s^2 + s\frac{R}{L} + \frac{1}{LC}\right)} \qquad (23)$$

Looking at the table of transform pairs, there are at least two – numbers 13 and 26 – which could be used to transform (23) back into the t-domain. I will use 26:

$$L^{-1}\left\{\frac{s+\alpha}{(s+a)^2 + b^2}\right\} = \frac{\sqrt{(\alpha - a)^2 + b^2}}{b}.e^{-at}\sin(bt + \phi) \qquad (24)$$

where

$\alpha = 0$

ϕ = atan2(b, α-a) = atan2(b, -a)

So:

$$V_L(t) = V.\left[\frac{\sqrt{(0-a)^2 + b^2}}{b} e^{-at} \sin(bt + \phi)\right]$$

$$V_L(t) = V.\left[\frac{\sqrt{a^2 + b^2}}{b} e^{-at} \sin(bt + \phi)\right]$$

$$V_L(t) = V.\left[\frac{\omega_n}{b} e^{-at} \sin(bt + \phi)\right]$$

$$V_L(t) = \frac{V\omega_n}{b} e^{-at} \sin(bt + \phi) \qquad (25)$$

Voltage across the inductor $V_L(t)$ – calculus method.

As for the capacitor, given that we already have an expression for i(t) – equation (1) above – we can also derive an expression for $V_L(t)$ using calculus rather than further use of Laplace Transforms.

For any inductor we can say that

$$V_L(t) = L\frac{di}{dt} \qquad (26)$$

and we also have

$$i(t) = \frac{V}{L}.\frac{1}{b}.e^{-at}\sin(bt) \qquad (1)$$

we therefore need to differentiate (1) to give di/dt then substitute this into (26) to derive the expression for $V_L(t)$.

since $\frac{d}{dt}(kt) = k\frac{d}{dt}(t)$

we can say that

$$\frac{di}{dt} = \frac{V}{Lb}.\frac{d}{dt}\left(e^{-at}.\sin(bt)\right) \qquad (27)$$

i.e. we need to find

$$\frac{d}{dt}\left(e^{-at}.\sin(bt)\right)$$

We have two terms in "t", so this is differentiation of a product i.e.

$$\frac{d}{dt}(uv) = v\frac{du}{dt} + u\frac{dv}{dt}$$

Let $u = e^{-at} \therefore \frac{du}{dt} = -ae^{-at}$

$v = \sin(bt) \therefore \frac{dv}{dt} = b\cos(bt)$

$$\therefore \frac{d}{dt}\left(e^{-at}.\sin(bt)\right) = -\sin(bt).ae^{-at} + e^{-at}.b\cos(bt)$$

$$\frac{d}{dt}\left(e^{-at}.\sin(bt)\right) = e^{-at}.b\cos(bt) - e^{-at}.a\sin(bt)$$

$$\frac{d}{dt}\left(e^{-at}.\sin(bt)\right) = e^{-at}\left(b\cos(bt) - a\sin(bt)\right) \qquad (28)$$

Substituting (28) into (27):

$$\frac{di}{dt} = \frac{Ve^{-at}}{Lb}\left(b\cos(bt) - a\sin(bt)\right) \qquad (29)$$

and substituting (29) into (26)

$$V_L(t) = \frac{LVe^{-at}}{Lb}\left(b\cos(bt) - a\sin(bt)\right)$$

$$V_L(t) = \frac{Ve^{-at}}{b}\left(b\cos(bt) - a\sin(bt)\right)$$

also

$$b\cos(bt) - a\sin(bt) = c\sin(bt + \phi)$$

where

$$c = \sqrt{\left(a^2 + b^2\right)}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

so

$$V_L(t) = \frac{V\sqrt{a^2 + b^2}}{b} e^{-at} \sin(bt + \phi)$$

$$V_L(t) = \frac{V\sqrt{a^2 + \omega_n^2 - a^2}}{b} e^{-at} \sin(bt + \phi)$$

$$V_L(t) = \frac{V\omega_n}{b} e^{-at} \sin(bt + \phi) \qquad (30)$$

which is the same result as (25), found by the Laplace method. Whilst the calculus method for V_L is not as involved as for V_C, I believe it is still more complex than the analysis using Laplace Transforms.

Conclusion.

In Part 1 I demonstrated how to derive an expression for i(t) for the series RLC circuit using the method of Laplace Transforms:

$$i(t) = \frac{V}{L}.\frac{1}{b}.\sin(bt) \qquad (1)$$

In part 2I went on to show how expressions for V_R, V_C and V_L could also be derived:

$$V_R(t) = \frac{VR}{L}.\frac{1}{b}.\sin(bt) \qquad (4)$$

$$V_C(t) = V\left[1 + \frac{\omega_n}{b}.e^{-at}\sin(bt - \phi)\right] \qquad (11)$$

$$V_L(t) = \frac{V\omega_n}{b}e^{-at}\sin(bt + \phi) \qquad (25)$$

where

ϕ = atan2(b, -a)

$$b = \sqrt{\omega_n^2 - a^2}$$

$$a = \frac{R}{2L}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

For V_C and V_L both the Laplace and calculus methods were used, in order to demonstrate the relative simplicity of the former compared to the latter.