
How to do it.

The process of analysing a circuit using the Laplace technique can be broken down into a series of straightforward steps:

1. Draw the circuit!
2. Replace each element in the circuit with its Laplace (s-domain) equivalent.
3. Apply Ohms law to the circuit or KVL/KCL if necessary
4. Rearrange the s-terms into one of the "standard" transform-pair forms and transform the result back into the time (t-) domain.
5. Verify your result.

Some of that may not make much sense at the moment, but will hopefully come clear as I work through the example. As with most things in life, the more you practice the easier it becomes, until using the Laplace technique becomes almost second nature.

1. Draw the circuit!

The circuit configuration which we’re going to analyse is shown in Figure 1.

![Figure 1 The circuit.](image)
In Figure 1, SW1 and SW2 are high-speed semiconductor switches. These could be discrete devices (in the case of, for example, a DC-DC converter) or switches integrated into the output stage of an IC of some description. SW1 and SW2 are never on at the same time, and either SW1 is turned on, connecting the load to Vcc or SW2 is turned on, connecting the load to 0V. Ringing is most likely to occur at the instant when current "commutates" between SW1 and SW2 or vice versa, see Figure 2.

Figure 2 Commutation and associated ringing

Considering Figure 2 (left), current flow commutates from SW2 to SW1 and positive ringing is seen at the mid-point of the two switches. In Figure 2 (right), current flow commutates from SW1 to SW2 and negative ringing is seen at the mid-point. In both cases, current flow just before commutation is shown by the dotted arrows and current flow after commutation is by the solid arrows.

You'll notice that I have drawn capacitors across the switches when they are open. This is because, even in the off-state, a semiconductor switch doesn't completely vanish from the circuit. There will always be some capacitance between its terminals, as shown, and this capacitance can play an important part in any ringing that may occur.

The analysis of the two commutations is identical, except for some sign changes, and so for simplicity I will just concentrate on the left-hand case, where commutation occurs from SW2 to SW1 and the ringing is positive.
We haven't finished analysing the circuit yet, though, as any real circuit will be made from conductive elements which also have resistance and inductance. We therefore need to redraw the circuit in order to take these elements into account. See Figure 3.

![Circuit Diagram](image)

**Figure 3** The circuit redrawn as a series-RLC configuration driven by a step change in voltage

In Figure 3, L is the combined inductance of all the various interconnecting bits of the circuit, R is their resistance (including the on-state resistance of SW1) and C is the off-state capacitance of switch SW2. We now have a "series RLC" circuit which is subject to a positive step change in voltage during commutation. Perfect conditions for ringing to occur!

2. Replace each element in the circuit with its Laplace (s-domain) equivalent.

Substitution of the (t-domain) circuit elements with their Laplace-(s-domain) equivalents is carried out according to the very simple rules summarized in Table 1 below.

<table>
<thead>
<tr>
<th>Description</th>
<th>t-domain</th>
<th>s-domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>Capacitor</td>
<td>C</td>
<td>1/sC</td>
</tr>
<tr>
<td>Inductor</td>
<td>L</td>
<td>sL</td>
</tr>
<tr>
<td>Positive step change in voltage</td>
<td>V</td>
<td>V/s</td>
</tr>
</tbody>
</table>

Table 1. t-domain circuit elements with their Laplace-(s-domain) equivalents summarized

I will add more entries into Table 1 in future analyses.
I won’t explain here why these substitutions can be made from t- to s-domain as, once again, this is not something which you need to know in order to use the technique. There are many explanations available on the web, with Sections 12 and 13 of this site demonstrating the transformations of capacitors and inductors.

Note also that I have assumed that the voltage V experiences an instantaneous step change from 0V to Vcc. Of course this would never be achievable in reality, although exceptionally fast rising and falling edges are common in many contemporary circuits. I will consider the case of a more realistic ramp change in voltage in one of my later analyses.

Now that we have a table of substitutions, we can make the appropriate additions to Figure 3. See Figure 4.

![Diagram of series RLC circuit with s-domain substitutions](image)

Figure 4 The series RLC circuit with s-domain substitutions

You will note also that I have included the current $i(s)$, where the “s” in brackets indicates current in the s-domain, as distinct from $i(t)$ (or more often than not just “i”) for current in the t-domain.

3. Apply Ohm’s law to the circuit.

One of the really useful aspects of Laplace circuit analysis is that once we have transformed the circuit elements into the s-domain, we can treat them in exactly the same manner as resistances and DC voltage and current sources in the t-domain. This principle extends to the use of Ohm’s law as well, so in the t-domain we have:

$$V = LR$$

and in the s-domain:

$$V(s) = i(s).Z(s)$$
where $Z(s)$ is the combined “Laplace impedance” of the circuit elements.

In our case, $Z(s)$ is the series combination of $R$, $L$ and $C$ and so for $Z(s)$ we have:

$$Z(s) = R + sL + \frac{1}{sC} \quad (3)$$

it really is that simple – just add them together as if they were resistors in series. And so the expression for $i(s)$ becomes:

$$i(s) = \frac{V(s)}{Z(s)} = \frac{V}{s} \cdot \frac{1}{R + sL + \frac{1}{sC}} = \frac{V}{s} \cdot \frac{s}{R + sL + \frac{1}{sC}} \quad (4)$$

where

$$V(s) = \frac{V}{s}$$

Now we have an expression for $i(s)$ we can move onto step 4.

4. **Rearrange the s-terms into one of the "standard" transform-pair forms and transform the result back into the time (t-) domain.**

When carrying out circuit analysis using Laplace Transforms, one of the most important resources to have to hand is a good table of Laplace Transform pairs. This table will have two columns: one column will be populated with expressions in terms of $s$ and the other will have the corresponding expressions in terms of $t$. These are transform pairs which have been worked out by many different people over time, using the fundamental definition of the Laplace Transform. Be thankful to those people – they did a lot of work so that you don’t have to! There are many such tables available in text books and on the internet, and in my opinion the most comprehensive and useful is at:

http://www.me.unm.edu/~starr/teaching/me380/Laplace.pdf

It will be helpful in understanding what follows if you take the time to follow this link. From this point onwards I will refer to transform pairs in this table by number.

Perhaps the most challenging part the Laplace analysis method is finding a standard pair which matches (or can be made to match) the s-domain expression which we have derived from our circuit. You will recall from (4) that we have:

$$i(s) = \frac{V}{s} \cdot \frac{s}{R + sL + \frac{1}{sC}}$$

is there a transform pair whose expression in $s$ looks anything like the right-hand side of (4)? At first glance, one might say “no”, but if we rearrange (4) just a little we have:
As $V$ and $L$ are constants, we can take the term $V/L$ out of the transformation process and concentrate solely on the term

$$ \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} $$

the denominator of this expression is a quadratic in $s$, so is there anything in the table that also looks like it has a quadratic in the denominator? The answer is yes – number 24 is of the form:

$$ L^{-1} \left[ \frac{1}{(s + a)^2 + b^2} \right] = \frac{1}{b} e^{-at} \sin(bt) $$

(the $L^{-1}$ notation simply means “the inverse Laplace transform of…” – it has nothing to do with reciprocals).

If we can extract expressions for $a$ and $b$ from (6) then success will be ours. Expanding out the denominator of the LHS of (7) and equating coefficients with the denominator of (6) we have:

$$ s^2 + 2as + (a^2 + b^2) \equiv s^2 + \frac{R}{L}s + \frac{1}{LC} $$

The coefficient of $s^2$ is 1 on both sides, so nothing further to do there.

For the coefficient of $s$ we have:

$$ 2a = \frac{R}{L} \therefore a = \frac{R}{2L} $$

(8)

And for the constant term:

$$ (a^2 + b^2) = \frac{1}{LC} \therefore b = \frac{1}{\sqrt{LC - \left(\frac{R}{2L}\right)^2}} $$

the term

$$ \frac{1}{\sqrt{LC}} $$

is usually referred to as the “resonant frequency” of the circuit, $\omega_n$, and so (9) may be more conveniently written as:

$$ b = \sqrt{\omega_n^2 - a^2} $$

(10)
We are almost there! We have everything we need to complete the RHS of (7) above: we have expressions for $a$ and $b$ so we can make the transformation back into the $t$-domain. Not forgetting of course the $V/L$ term we took out of the transformation process, we now have:

$$i(t) = \frac{V}{L} \cdot \frac{1}{\sqrt{\omega_n^2 - a^2}} \cdot \sin\left(\sqrt{\omega_n^2 - a^2} t\right)$$  \hspace{1cm} (11)

remembering that $i(t)$ simply means $I$ in the $t$- (or time-) domain.

I have to say that the first time I produced this result I was absolutely ecstatic. I think I may have even cheered. I had no idea what the expression for $i(t)$ would look like or even if the method would work at all, but even without verification this expression looked right.

5. Verify your result.

There are a few different methods we can use to check the validity of (11). The first is to check the “dimensions” of the equation. What does that mean?

Every physical quantity has a unit e.g. ohms, henries, volts and so on. These units are derived from the seven basic SI Base Units of the metre (m), kilogram (kg), second (s), ampere (A), Kelvin (K), candela (cd) and mole (mol). Any unit (ohm, henry, etc.) can be expressed in terms of the Base Units, e.g.

- ohm $[\text{Kg}.m^2.A^{-2}.s^{-3}]$
- henry $[\text{Kg}.m^2.A^{-2}.s^{-2}]$
- farad $[\text{Kg}^{-1}.m^2.A^2.s^4]$ 
- volt $[\text{Kg}.m^2.A^{-1}.s^3]$ 

In order for an equation to be correct, the quantities on both sides of the “$=$” sign must have the same units. In addition, if we wish to add or subtract quantities then they too must have the same units. Note that in this context, “s” is the unit of time – seconds – not the Laplace operator.

In (11) the term

$$\sqrt{\omega_n^2 - a^2}$$ 

appears twice, so a good place to start would be in determining whether the quantities $\omega_n^2$ and $a^2$ have the same units. If they don’t then clearly we have made a mistake and the equation is incorrect. From (8) and (9) we know that:

$$\sqrt{\omega_n^2 - a^2} = \left(\frac{1}{\sqrt{LC}} - \frac{R}{2L}\right)^2$$

Expressing the two quantities in Base Units we have:
\[
\frac{1}{LC} = \left[ \frac{1}{Kg.m^2.A^{-2}.s^{-2}} \right] \left[ Kg^{-1}.m^{-2}.A^2.s^4 \right] = s^{-2}
\]

\[
\left( \frac{R}{2L} \right)^2 = \left( \frac{[Kg.m^2.A^{-2}.s^{-3}]}{2[Kg.m^2.A^{-2}.s^{-2}]} \right)^2 = \frac{s^{-2}}{2}
\]

so the units of both quantities match and we can subtract them. The units of \(\sqrt{\omega_n^2 - a^2}\) are therefore \(s^{-1}\), and the quantity \(\left( \frac{\sqrt{\omega_n^2 - a^2}}{t} \right)\) is dimensionless as the dimensions of \(t\) are \(s\) (seconds). \(\sin\left( \frac{\omega_n t}{a} \right)\) is also dimensionless.

Examining the dimensions for the \(V/L\) term of (11) we can see that:

\[
i(t) = \left[ \frac{Kg.m^2.A^{-1}.s^3}{Kg.m^2.A^{-2}.s^{-2}} \right] \cdot s^{-1} = \frac{1}{A^{-1}} = A
\]

i.e. the unit of \(i(t)\) is amps – exactly what it should be.

This exercise is a useful sanity-check which at least tells us that the answer isn’t wrong, even though it cannot definitely tell us that the answer is right.

A second useful verification we can make is to reproduce the circuit of Figure 3 in LTSPICE with appropriate component values, run the simulation and plot the resulting waveform of \(i(t)\). If we then calculate values of \(i(t)\) for various values of \(t\) using (11) and a spreadsheet then we should find that the two plots match exactly. A screenshot of the LTSPICE simulation is shown in Figure 5.
Figure 5 LTSPICE simulation of the series RLC circuit (R = 1Ω, L = 5nH, C = 200pF)

A comparison of the LTSPICE and spreadsheet results is shown in Figure 6.
Figure 6 Comparison of the LTSPICE and spreadsheet results
The result are the same! Given that SPICE programs use a completely different method of analysis to the Laplace method, I’d say that this is a pretty good vindication of the results produced using the Laplace method. If I had the resources, I would now go on to try to validate the results using a real-life circuit, but sadly that option is not open to me at the moment. I hope the comparison with the LTSPICE simulation results is enough to convince you.

Final words.

I hope this exercise has given you a brief introduction to the use of Laplace transforms, and how they can be used to analyse a simple circuit without being at all scary. I had hoped to show how expressions for the voltages across the components could also be derived, but I think this document is already long enough. Finding expressions for $V_R$, $V_L$ and $V_C$ is a topic I’ll come on to in Part 2.

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