5: ROOTS OF A QUADRATIC EQUATION

The general form of a quadratic equation

We have grown accustomed to recognising a quadratic equation in the form \( ax^2 + bx + c = 0 \). In this section, we will be introduced to a new format for such a quadratic equation. This format would express the quadratic in the form of its roots. It is a convenient form to know and it allows us the flexibility to switch from this form to the standard form.

Roots of a quadratic equation (\( \alpha \text{ and } \beta \))

A quadratic equation in \( x \) is of the general form \( ax^2 + bx + c = 0 \), where \( a, b \) and \( c \) are constants.

If we divide each term by \( a \), then the quadratic equation can be expressed in an equivalent form with the coefficient of \( x^2 \) is equal to one as shown below.

\[
ax^2 + bx + c = 0
\]
\[
\frac{x^2}{a} + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{(1)}
\]

Now consider \( \alpha \) and \( \beta \) as the roots of the quadratic.

We can now rewrite the quadratic in the form:

\[
(x - \alpha)(x - \beta) = 0.
\]

By expanding we get,

\[
x^2 - (\alpha + \beta)x + \alpha\beta = 0. \quad \text{(2)}
\]

Equation (2) is an equivalent form of equation (1). In fact, any quadratic equation, in \( x \), can always be expressed in the form of its roots.

We can replace \((\alpha + \beta)\) by the ‘sum of the roots’ and \(\alpha\beta\) by the ‘product of the roots’, to obtain the following form for a quadratic equation.

\[
x^2 - \text{(sum of roots)}x + \text{product of roots} = 0
\]

Sum and product of the roots of a quadratic equation

Equations (1) and (2) above are two equivalent forms of a quadratic equation.

Equating both forms we get:

\[
x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha + \beta)x + \alpha\beta
\]

When we equate coefficients, the following is obtained:

\[
\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.
\]

We can now make a general statement about the roots of a quadratic.

For the quadratic equation \( ax^2 + bx + c = 0 \),
the sum of the roots \( \frac{-b}{a} \) and
the product of the roots \( \frac{c}{a} \).

Example 1

If \( \alpha \) and \( \beta \) are the roots of the quadratic equation \( x^2 - 3x + 2 = 0 \), determine

(i) the sum of the roots and
(ii) the product of the roots.

Solution

In the quadratic equation \( x^2 - 3x + 2 = 0 \)
\( a = 1, b = -3 \) and \( c = 2 \).

(i) The sum of the roots, \( \alpha + \beta = \frac{-(-3)}{1} = 3 \)

(ii) The product of the roots, \( \alpha\beta = \frac{2}{1} = 2 \).

Example 2

The quadratic equation \( x^2 - 4x + 3 = 0 \) has roots \( \alpha \) and \( \beta \).

a) Obtain the equation whose roots are \( \alpha + 1 \) and \( \beta + 1 \).

b) Obtain the equation whose roots are \( \alpha^2 \) and \( \beta^2 \).

Solution

If the equation \( x^2 - 4x + 3 = 0 \) has roots \( \alpha \) and \( \beta \),
then \( a = 1, b = -4 \) and \( c = 3 \). Hence,

\( \alpha + \beta = 4 \) and \( \alpha\beta = 3 \)

To obtain an equation whose roots are \( \alpha + 1 \) and
\[ \beta + 1, \text{ we can substitute these roots in the following equation:} \]
\[ x^2 - (\text{sum of roots}) x + \text{product of roots} = 0 \]
\[ x^2 - [(\alpha + 1) + (\beta + 1)]x + [(\alpha + 1)(\beta + 1) = 0 \]
\[ x^2 - [(\alpha + \beta + 2)]x + [\alpha \beta + (\alpha + \beta) + 1] = 0 \]
\[ x^2 - (4 + 2)x + (3 + 4 + 1) = 0 \]
\[ x^2 - 6x + 8 = 0 \]

This is the required equation.

Part b) To obtain an equation whose roots are \( \alpha \) and \( \beta \), we substitute these roots in:
\[ x^2 - (\alpha^2 + \beta^2) x + (\alpha^2 \times \beta^2) = 0 \]
\[ x^2 - (\alpha^2 + \beta^2) x + (\alpha^2 \beta^2) = 0 \]

[Recall: \((\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta\)]
\[ x^2 - (\alpha^2 + \beta^2 - 2\alpha\beta)x + (\alpha^2 \beta^2) = 0 \]
\[ x^2 - (4^2 - 2(3)) x + (3)^2 = 0 \]
\[ x^2 - 10x + 9 = 0 \]

This is the required equation.

**Example 3**

Given that \( x^2 + (k - 5)x - k = 0 \) has real roots which differ by 4, determine

i. the value of each root

ii. the value of \( k \).

**Solution**

If we let \( \alpha \) be the smaller real root, then the other will be \((\alpha + 4)\).

Then the sum of the roots is : \( \alpha + (\alpha + 4) = 2\alpha + 4 \)
The product of the roots is \( \alpha(\alpha+4) \).

From the given equation \( x^2 + (k - 5)x - k = 0 \),
The sum of the roots is: \(- (k - 5)\)
The product of the roots is: \(-k\)

Equating coefficients, we have:

<table>
<thead>
<tr>
<th>Sum of roots</th>
<th>Product of roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2\alpha + 4 = -(k - 5) )</td>
<td>( \alpha(\alpha+4) = -k )</td>
</tr>
<tr>
<td>( 2\alpha + 4 = -k + 5 )</td>
<td>( k = -\alpha(\alpha + 4) )</td>
</tr>
<tr>
<td>( k = 1 - 2\alpha )</td>
<td>( \alpha = 1 - 2 \alpha )</td>
</tr>
</tbody>
</table>

Equating equations (1) and (2) to eliminate \( k \), we have:

\[ -\alpha^2 - 4 \alpha = 1 - 2 \alpha \]
\[ \alpha^2 + 2 \alpha + 1 = 0 \]
\[ (\alpha + 1)(\alpha + 1) = 0 \]
\[ \alpha = -1 \]

The value of \( k \): \( k = 1 - 2\alpha = 1 - 2(-1) = 3 \)
\[ \therefore \text{Roots are} -1 \text{ and } -1 + 4 \]
The roots are -1 and 3.

**Alternative Method**

If we let \( \alpha \) be the smaller real root, then the other will be \((\alpha + 4)\).

Hence the quadratic equation may be expressed as
\[ (x - \alpha)[x - (\alpha + 4)] = 0 \]
\[ \alpha x - ax + \alpha^2 - 4x + 4\alpha = 0 \]
\[ x^2 + (-2\alpha - 4)x + (\alpha^2 + 4\alpha) = 0 \]

Equating coefficient of \( x \), we obtain
\[ -2\alpha - 4 = k - 5 \]
\[ -2\alpha - 4 = k - 5 \]
\[ k = 1 - 2\alpha \]
\[ \therefore \alpha = \frac{1-k}{2} \]

Equating constant terms, we obtain
\[ \alpha^2 + 4\alpha = -k \]
\[ \therefore \left(\frac{1-k}{2}\right)^2 + 4\left(\frac{1-k}{2}\right) = -k \]
\[ 1 - 2k + k^2 = 2 - 2k + k = 0 \]
\[ 1 - 2k + k^2 + 8 - 8k + 4k = 0 \]
\[ k^2 - 6k + 9 = 0 \]
\[ (k - 3)^2 = 0 \]
\[ k = 3 \]

When, \( k = 3 \), \( \alpha = \frac{1-3}{2} = -1 \)
\[ \therefore \text{Roots are} -1 \text{ and } -1 + 4 \]
The roots are -1 and 3.