22. VARIATION

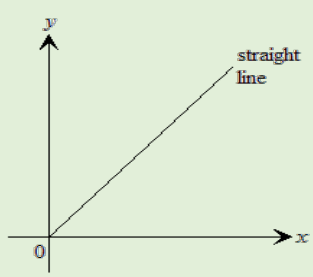
DIRECT AND INVERSE VARIATION

In our study of functions, we represented relationships between two variables using a rule. We examined rules such as $y = 2x$ or $y = 3x^2$, where x and y are variables. In this chapter, we will differentiate between two types of relationships - direct and inverse relationships.

Direct Variation

If two quantities vary such that as one increases the other also increases *by the same factor*, then the quantities are said to be in direct variation or a direct proportionality.

Letting x and y represent these quantities, we can write this relationship in symbol form as $y \propto x$, read as 'y is directly proportional to x'. When this is so, the ratio of $y : x$ is always a constant, A graph of y against x will therefore yield a straight line whose gradient is k and which passes through the origin. This constant is called the constant of variation or the constant of proportionality.

Direct Proportion	Graphical representation
$y \propto x$ $\frac{y}{x} = k$ and $y = kx$, where k is the constant of variation.	

To determine, k, the constant of variation we must have at least one pair of corresponding values of the variables. For example, the table below displays three sets of corresponding values of two variables, x and y which are in direct proportion.

x	2	4	7
y	6	12	21

The ratios of corresponding values are:

$$\frac{y}{x} = \frac{6}{2} = \frac{12}{4} = \frac{21}{7} = 3.$$

The constant of variation, $k = 3$ and this will also be the gradient of the line $y = 3x$. Notice that we could have selected any pair of corresponding values to determine the value of k.

Example 1

Given that A varies directly as r and $A = 8$ when $r = 32$,
(i) find k, the constant of variation
(ii) the value of A when $r = 80$.

Solution

(i) We write A varies directly as r as

$$A \propto r$$
$$\text{Hence, } A = kr$$
$$8 = k \times 32$$
$$k = 8 \div 32 = \frac{1}{4}$$

(ii) We can substitute the value of k in $A = kr$ to obtain

$$A = \frac{1}{4}r$$

When $r = 80$,

$$A = \frac{1}{4}(80)$$
$$A = 20$$

In direct variation, it is also possible to have other examples where the variables involve powers, roots or any other function. For example,

y varies directly as the square of x, that is

$$y \propto x^2$$

$y = kx^2$ (where k is the constant of proportion)

In this case, a graph of y against x^2 will produce a straight line of gradient k and which passes through O.

y varies directly as the square root of x, that is

$$y \propto \sqrt{x}$$

$y = k\sqrt{x}$, (where k is the constant of proportion)
A graph of y against $\sqrt{x}$ will produce a straight line of gradient k and which passes through O.

y^2 varies directly as the cube of x, that is

$$y^2 \propto x^3$$
$$y^2 = k\sqrt[3]{x}$$

A graph of y^2 against $\sqrt[3]{x}$ will produce a straight line of gradient k and which passes through O.

In all the above cases, the relationship is still linear, because the constant, k, is still the gradient of a straight line.

Example 2

Given that the square of V is proportional to the cube of t and that $V = 3$ when $t = 2$, find k, the constant of variation.

Solution

We write V is proportional to the cube of t as

$$V^2 \propto t^3$$

$$Hence\ V^2 = kt^3$$

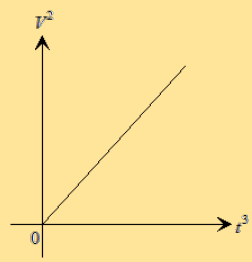
(where k is the constant of proportion)

Since $V = 3$ when $t = 2$,

$V^2 = kt^3$

$3^2 = k \times 2^3$

$k = \frac{9}{8}$



The graph of V^2 against t^3 will produce a straight line.

Example 3

The root of P varies directly as the square of Q. Given that $P = 64$ when $Q = 3$, calculate
(i) k, the constant of variation
(ii) the value of P when $Q = 6$.

Solution

(i) The root of P varies directly as the square of Q is written as:

$$\Rightarrow \sqrt{P} \propto Q^2$$

$$\therefore \sqrt{P} = kQ^2$$

$$\sqrt{P} = kQ^2$$

$$\sqrt{64} = k \times 4^2$$

$$k = \frac{8}{16}$$

$$k = \frac{1}{2}$$

(ii) the value of P when $Q = 6$.

$$\sqrt{P} = kQ^2$$

$$\sqrt{P} = \frac{1}{2}Q^2$$

$$\sqrt{P} = \frac{1}{2} \times 6^2$$

$$\sqrt{P} = \frac{1}{2} \times 36$$

$$\sqrt{P} = 18$$

$$P = 18^2$$

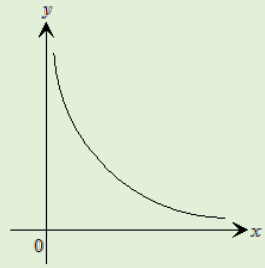
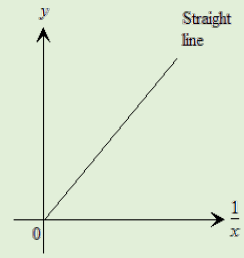
$$P = 324$$

In the above problem, a graph of $\sqrt{P}$ against Q^2 will produce a straight line and which passes through O. However, in formulating the solutions to variation problems, it is not necessary to sketch the graph unless requested to do so.

Inverse Variation

If two quantities vary in such a way that as one increases the other decreases *by the same factor*, then the quantities are said to be in inverse or indirect variation.

Letting x and y represent the quantities, we can write this relationship in symbol form as $y \propto \frac{1}{x}$, read as 'y is inversely proportional to x'. A graph of y against x will yield a curve. However, as expected, the graph of y against $\frac{1}{x}$ will yield a straight line since y is actually proportional to $\frac{1}{x}$.

Inverse Variation	Graphical Representation
$y \propto \frac{1}{x}$ $y = \frac{k}{x}$ $yx = k$ where k is the constant of variation.	
 $y\ versus\ x$	 $y\ versus\ \frac{1}{x}$

The table below displays a table of values for the variables, x and y that are in inverse proportion. Notice that as x increases, y decreases.

x	1	2	3
y	15	7.5	5

We may use and set of corresponding values to obtain say $1 = \frac{k}{15}$. So $k = 15$

Any other pair of values would have yielded the same value for k.

Like direct variation, we can also have examples of indirect variation where the variables involve powers, roots or any other function, such as:

y varies indirectly as the square of x, that is

$$y \propto \frac{1}{x^2}$$

A graph of y against $\frac{1}{x^2}$ will produce a straight line of gradient k and passing though O.

y varies indirectly as the square root of x,

$$y \propto \frac{1}{\sqrt{x}}$$

A graph of y against $\frac{1}{\sqrt{x}}$ will produce a straight line of gradient k and passing though O.

y^2 varies indirectly as the cube root of x,

$$y \propto \frac{1}{\sqrt[3]{x}}$$

A graph of y against $\frac{1}{\sqrt[3]{x}}$ will produce a straight line of gradient k and passing though O.

Example 4

Given that y varies inversely as x and that $y = 4$ when $x = 3$, determine
(i) the constant of variation, k
(ii) the value of y when $x = 6$.

Solution

(i) y varies inversely as x is written as

$y \propto \frac{1}{x}$

$y = \frac{k}{x}$

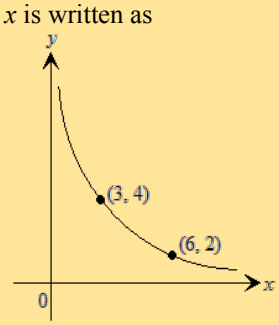
$y = 4$ when $x = 3$

$4 = \frac{k}{3}$

$k = 12$

Substituting for k:

$y = \frac{12}{x}$



(ii) When $x = 6, y = \frac{12}{6} = 2$

A sketch of the graph of y against x is shown above. We obtain a curve because we plotted y against x.

If we ploy y against $\frac{1}{x}$ we will get a straight line.

Example 5

Given that the square of y is inversely proportional to the root of x and $y = 5$ when $x = 4$, find y when $x = 81$.

Solution

The square of y is inversely proportional to the root of x is written as:

$$y^2 \propto \frac{1}{\sqrt{x}}$$

$$y^2 = \frac{k}{\sqrt{x}}$$

$y = 5$ when $x = 4$

$$5^2 = \frac{k}{\sqrt{4}}$$

$$k = 25 \times 2 = 50$$

Substituting for k:

$$y^2 = \frac{50}{\sqrt{x}}$$

When $x = 81$

$$y^2 = \frac{50}{\sqrt{81}} = \frac{50}{9} = 5.56$$

$$y = \sqrt{5.56} = 2.36$$

Example 6

Given that the speed, s kilometres per hour of an aircraft is inversely proportional to its wingspan, W, in metres and that $s = 600$ when $W = 75$.
Calculate
(i) the value of s when $W = 100$
(ii) the value of W when $s = 400$

Solution

(i) The speed, s is inversely proportional to its wingspan, W is written as

$s \propto \frac{1}{W}$

$s = \frac{k}{W}$

$600 = \frac{k}{75}$

$k = 600 \times 75$

$k = 45000$

Substitute for k, $\mathbf{s = \frac{45000}{W}}$,

Hence, $\mathbf{s = \frac{45000}{100} = 450}$

(ii) To calculate the value of W when $s = 400$

$s = \frac{45000}{w}$

$400 = \frac{45000}{W}$

$W = \frac{45000}{400}$

$W = 112.5$